

SPACES OF CURVES WITH CONSTRAINED CURVATURE ON HYPERBOLIC SURFACES

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ABSTRACT. Let S be a hyperbolic surface. We investigate the topology of the space of all curves on S which start and end at given points in given directions, and whose curvatures are constrained to lie in a given interval (κ_1, κ_2) . Such a space falls into one of four qualitatively distinct classes, according to whether (κ_1, κ_2) contains, overlaps, is disjoint from, or contained in the interval $[-1, 1]$. Its homotopy type is computed in the latter two cases. We also study the behavior of these spaces under covering maps when S is arbitrary (not necessarily hyperbolic nor orientable) and show that if S is compact then they are always nonempty.

0. INTRODUCTION

A curve on a surface S is said to be *regular* if it has a continuous and nonvanishing derivative. Consider two such curves whose curvatures take values in some given interval, starting in the same direction (prescribed by a unit vector tangent to S) and ending in another prescribed direction. It is a natural problem to determine whether one curve can be deformed into the other while keeping end-directions fixed and respecting the curvature bounds. From another viewpoint, one is asking for a characterization of the connected components of the space of all such curves. More ambitiously, what is its homotopy or homeomorphism type? The answer can be unexpectedly interesting, and it is closely linked to the geometry of S .

In this article this question is investigated when S is hyperbolic, that is, a (possibly nonorientable) smooth surface endowed with a complete Riemannian metric of constant negative curvature, say, -1 . It is well known that any such surface can be expressed as the quotient of the hyperbolic plane \mathbb{H}^2 by a discrete group of isometries. Thus any curve on S can be lifted to a curve in \mathbb{H}^2 having the same curvature (at least in absolute value, if S is nonorientable). As will be more carefully explained later, this implies that one can obtain a solution of the problem about spaces of curves on S if one knows how to solve the corresponding problem in the hyperbolic plane for all pairs of directions.

Let $u, v \in U\mathbb{H}^2$ (the unit tangent bundle of \mathbb{H}^2) be two such directions. Let $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ be the set of all smooth regular curves on \mathbb{H}^2 whose initial (resp. final) unit tangent vectors equal u (resp. v) and whose curvatures take values inside (κ_1, κ_2) , furnished with the C^∞ -topology. There are canonical candidates for the position of connected component of this space, defined as follows. Let

$$\text{pr}: \widetilde{U\mathbb{H}^2} \rightarrow U\mathbb{H}^2$$

denote the universal cover. Fixing a lift \tilde{u} of u , one obtains a map $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v) \rightarrow \text{pr}^{-1}(v)$ by looking at the endpoint of the lift to the universal cover, starting at \tilde{u} , of curves in $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$. Since $\text{pr}^{-1}(v)$ is discrete, this yields a decomposition of $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ into closed-open subspaces.

More concretely, let $\gamma: [0, 1] \rightarrow \mathbb{C}$ be a regular plane curve. An *argument* $\theta: [0, 1] \rightarrow \mathbb{R}$ for γ is a continuous function such that $\dot{\gamma}$ always points in the direction of $e^{i\theta}$; note that there are countably many such functions, which differ by multiples of 2π . The *total turning* of γ is defined as $\theta(1) - \theta(0)$. Because \mathbb{H}^2 is diffeomorphic to \mathbb{C} , any regular curve in the former also admits arguments and a total turning. However, these have no geometric meaning since they depend on the choice of diffeomorphism. In any case, once such a choice has been made, it gives rise to the decomposition

$$(1) \quad \mathcal{C}_{\kappa_1}^{\kappa_2}(u, v) = \bigsqcup_{\tau} \mathcal{C}_{\kappa_1}^{\kappa_2}(u, v; \tau),$$

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where $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v; \tau)$ consists of those curves in $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ which have total turning τ , and τ runs over all *valid* total turnings, viz., those for which v is parallel to $e^{i\tau}u$ (regarded as vectors in \mathbb{C}). The closed-open subspaces appearing on the right side of (1), which will be referred to as the *canonical subspaces* of $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$, are independent of the diffeomorphism: they are in bijective correspondence with the fiber $\text{pr}^{-1}(v)$. If $\kappa_1 = -\infty$ and $\kappa_2 = +\infty$, so that no restriction is imposed on the curvature, then they are in fact precisely the components of $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$, and each of them is contractible. However, for general curvature bounds they may fail to be connected, contractible, or even nonempty. The possibilities depend above all upon the relation of (κ_1, κ_2) to the interval $[-1, 1]$.

If (κ_1, κ_2) is contained in $[-1, 1]$, then at most one of the canonical subspaces of $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ is nonempty. This subspace is always contractible. Thus $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ itself is either empty or contractible.

If (κ_1, κ_2) is disjoint from $[-1, 1]$, then infinitely many of the canonical subspaces are empty, and infinitely many are nonempty. The latter are all contractible.

If (κ_1, κ_2) contains $[-1, 1]$, then none of the canonical subspaces is empty. We conjecture that most are contractible, but finitely many of them may have the homotopy type of an n -sphere, for some $n \in \mathbb{N}$ depending upon the subspace. This conjecture is partly motivated by results about the corresponding spaces of curves in the Euclidean plane obtained in [20] and [21].

Finally, if (κ_1, κ_2) overlaps $[-1, 1]$, then infinitely many of the canonical subspaces are empty, and infinitely many are nonempty. Nevertheless, as in the previous case, these subspaces may not be connected. We hope to determine the homotopy type in these two last cases in a future paper.

Outline of the sections. After briefly introducing some notation and definitions, §1 begins with a discussion of curves of constant curvature κ in the hyperbolic plane: circles ($|\kappa| > 1$), horocycles ($|\kappa| = 1$) and hypercycles ($|\kappa| < 1$). Then a transformation is defined which takes a curve and shifts it by a fixed distance along the direction prescribed by its normal unit vectors. Its effect on the regularity and curvature of the original curve is investigated, and applied to reduce the dependence of the topology of $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ to four real parameters, instead of the eight needed to specify κ_1 , κ_2 , u and v .

In §2 the voidness of the canonical subspaces is discussed. It is proven that if (κ_1, κ_2) is contained in $[-1, 1]$, then the image of any curve in $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ is the graph of a function when seen in an appropriate conformal model of the hyperbolic plane, which we call the Mercator model.¹ In particular, at most one of its canonical subspaces is nonempty. If (κ_1, κ_2) contains $[-1, 1]$, then none of the canonical subspaces is empty. In the two remaining cases, there is a critical value τ_0 for the total turning such that $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v; \tau)$ is nonempty for all $\tau \geq \tau_0$ and empty for all $\tau < \tau_0$ (or the reverse, depending on whether (κ_1, κ_2) contains points to the right or to the left of $[-1, 1]$).

Section 3 explains how a curve of constrained curvature may also be regarded as a curve in the group of orientation-preserving isometries of \mathbb{H}^2 satisfying certain conditions. This perspective is sometimes useful.

In §4 it is shown that the nonempty canonical subspaces of $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ are all contractible when (κ_1, κ_2) is disjoint from $[-1, 1]$. The idea is to parametrize all curves in such a subspace by the argument of its unit tangent vector when viewed as a curve in the half-plane model, and to take (Euclidean) convex combinations. The proof intertwines various Euclidean and hyperbolic concepts, and seems to be highly dependent on this particular model.

In the Mercator model M , any curve in $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ may be parametrized as $x \mapsto (x, y(x))$ when (κ_1, κ_2) is contained in $[-1, 1]$, so the bounds on its curvature can be translated into two differential inequalities involving \dot{y} and \ddot{y} , but not y itself, because vertical translations are isometries of M . This allows one to produce a contraction of the space by working with the associated family of functions \dot{y} , as carried out in §5.

In §6 we define spaces of curves with constrained curvature on a general surface S , not necessarily hyperbolic, complete nor orientable, and explain how a Riemannian covering of S induces homeomorphisms between spaces of curves on S and on the covering space. We also show that if S is compact then any such space is nonempty; this is also true in the Euclidean plane, but not in the hyperbolic

¹It is very likely that the Mercator model has already appeared under other (possibly standard) names in the literature; however, we do not know any references. Hypercycles are also known as “hypercircles”.

plane, as mentioned previously. The section ends with a brief discussion of spaces of closed curves without basepoint conditions.

Several useful constructions, notably one which is essential to the proof of the main result of §5, may create discontinuities of the curvature, and thus lead out of the class of spaces $\mathcal{CS}_{\kappa_1}^{\kappa_2}(u, v)$ defined in §6. To circumvent this, the paraphernalia of L^2 functions is used in §7 to define another family of spaces, denoted $\mathcal{LS}_{\kappa_1}^{\kappa_2}(u, v)$, which are Hilbert manifolds. The curves in these spaces are regular but their curvatures are only defined almost everywhere. The main result of §7 states that the natural inclusion $\mathcal{CS}_{\kappa_1}^{\kappa_2}(u, v) \hookrightarrow \mathcal{LS}_{\kappa_1}^{\kappa_2}(u, v)$ is a homotopy equivalence with dense image. Sections 6 and 7 are independent of the other ones.

There is a considerable literature on the topology of spaces of curves subject to a constraint on their curvature (resp. torsion), especially when the constraint is that it should always be nonzero. We mention only [1, 2, 6, 7, 8, 10, 11, 12, 13, 14, 17, 18, 19, 20, 21, 22, 23].

A few exercises are included in the article. These are never used in the main text, and their solutions consist either of straightforward computations or routine extensions of arguments presented elsewhere.

It is assumed that the reader is familiar with the geometry of the hyperbolic plane as discussed, for instance, in chapter 2 of [25], the expository article [5], chapter 7 of [3] or chapters 3–5 of [16].

1. BASIC DEFINITIONS AND RESULTS

When speaking of the hyperbolic plane with no particular model in mind, we denote it by \mathbb{H}^2 . The underlying sets of the (Poincaré) disk, half-plane and hyperboloid models are denoted by:

$$\begin{aligned} D &= \{z \in \mathbb{C} \mid |z| < 1\}; \\ H &= \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}; \\ L &= \{(x_0, x_1, x_2) \in \mathbb{E}^{2,1} \mid -x_0^2 + x_1^2 + x_2^2 = -1 \text{ and } x_0 > 0\}. \end{aligned}$$

The circle at infinity is denoted by \mathbb{S}_∞^1 or $\partial\mathbb{H}^2$, the norm of a vector v tangent to \mathbb{H}^2 by $|v|$, and the Riemannian metric by $\langle \cdot, \cdot \rangle$. When working with D or H , both \mathbb{H}^2 and its tangent planes are regarded as subsets of \mathbb{C} .

In the disk and half-plane models, we will select the orientation which is induced on the respective underlying sets by the standard orientation of \mathbb{C} . In the hyperboloid model, a basis (u, v) of a tangent plane is declared positive if (u, v, e_0) is positively oriented in \mathbb{R}^3 ; equivalently, the Lorentzian vector product $u \otimes v$ points to the exterior region bounded by L (i.e., the one containing the light-cone).

Given a regular curve $\gamma: [0, 1] \rightarrow \mathbb{H}^2$, its *unit tangent* is the map

$$\mathbf{t} = \mathbf{t}_\gamma: [0, 1] \rightarrow UT\mathbb{H}^2, \quad \mathbf{t} := \frac{\dot{\gamma}}{|\dot{\gamma}|}.$$

Let $J: T\mathbb{H}^2 \rightarrow T\mathbb{H}^2$ denote the bundle map (“multiplication by i ”) which associates to $v \neq 0$ the unique vector Jv of the same norm as v such that (v, Jv) is orthogonal and positively oriented. Then the *unit normal* to γ is given by

$$\mathbf{n} = \mathbf{n}_\gamma: [0, 1] \rightarrow UT\mathbb{H}^2, \quad \mathbf{n} := J \circ \mathbf{t}.$$

Assuming that γ has a second derivative, its *curvature* is the function

$$(2) \quad \kappa = \kappa_\gamma: [0, 1] \rightarrow \mathbb{R}, \quad \kappa := \frac{1}{|\dot{\gamma}|} \left\langle \frac{D\mathbf{t}}{dt}, \mathbf{n} \right\rangle = \frac{1}{|\dot{\gamma}|^2} \left\langle \frac{D\dot{\gamma}}{dt}, \mathbf{n} \right\rangle;$$

here D denotes covariant differentiation (along γ). The hyperboloid model is usually the most convenient one for carrying out computations, since it realizes \mathbb{H}^2 as a submanifold of the vector space $\mathbb{E}^{2,1}$. For instance, the curvature of a curve γ on L is given by:

$$\kappa = \frac{\dot{\mathbf{t}} \cdot \mathbf{n}}{\|\dot{\gamma}\|} = \frac{\ddot{\gamma} \cdot \mathbf{n}}{\|\dot{\gamma}\|^2},$$

where \cdot denotes the bilinear form on $\mathbb{E}^{2,1}$ and $\|\cdot\|^2$ is the associated quadratic form.

1.1 Definition (spaces of curves in \mathbb{H}^2). Let $u, v \in UT\mathbb{H}^2$ and $\kappa_1 < \kappa_2 \in \mathbb{R} \cup \{\pm\infty\}$. Then $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ (resp. $\bar{\mathcal{C}}_{\kappa_1}^{\kappa_2}(u, v)$) denotes the set of C^r regular curves $\gamma: [0, 1] \rightarrow \mathbb{H}^2$ satisfying:

- (i) $\mathbf{t}_\gamma(0) = u$ and $\mathbf{t}_\gamma(1) = v$;
- (ii) $\kappa_1 < \kappa_\gamma < \kappa_2$ (resp. $\kappa_1 \leq \kappa_\gamma \leq \kappa_2$) throughout $[0, 1]$.

This set is furnished with the C^r topology, for some $r \geq 2$.²

1.2 Definition (osculation). Two curves $\gamma, \eta: [0, 1] \xrightarrow{C^2} S$ on a smooth surface S will be said to *osculate* each other at $t = t_0, t_1 \in [0, 1]$ if one may reparametrize γ , keeping t_0 fixed, so that

$$\gamma(t_0) = \eta(t_1) \in S, \quad \dot{\gamma}(t_0) = \dot{\eta}(t_1) \in TS \quad \text{and} \quad \ddot{\gamma}(t_0) = \ddot{\eta}(t_1) \in T(TS).$$

Remark. Suppose that S is oriented and furnished with a Riemannian metric. Then γ and η osculate each other at $t = t_0, t_1$ if and only if

$$\gamma(t_0) = \eta(t_1), \quad \mathbf{t}_\gamma(t_0) = \mathbf{t}_\eta(t_1) \quad \text{and} \quad \kappa_\gamma(t_0) = \kappa_\eta(t_1).$$

1.3 Definition (circle, hypercycle, horocycle, ray). A *circle* is the locus of all points a fixed distance away from a certain point in \mathbb{H}^2 . An *hypercycle* is one component of the locus of all points a fixed distance away from a certain geodesic in \mathbb{H}^2 . A *horocycle* is a curve which meets all geodesics through a certain point of $\partial\mathbb{H}^2$ orthogonally. A *ray* is a distance-preserving map $\alpha: [0, +\infty) \rightarrow \mathbb{H}^2$; such a ray is said to *emanate from* $\alpha'(0) \in U\mathbb{H}^2$.

In order to understand the effect of a geometric transformation on a given curve, it is often sufficient to replace the latter by its family of osculating constant-curvature curves. Because the hyperbolic plane is homogeneous and isotropic, one can represent these in a convenient position in one of the models as a means of avoiding calculations.

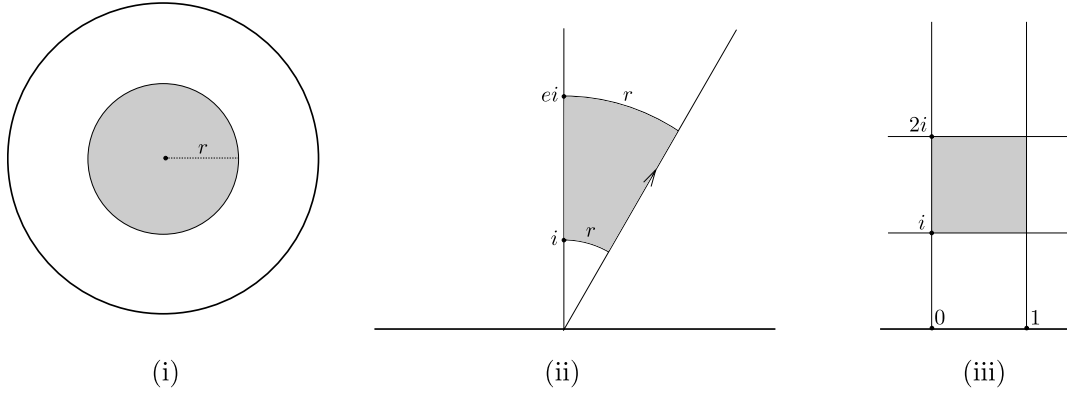


FIGURE 1. Computing the curvature of circles, hypercycles and horocycles through Gauss-Bonnet. The circle in (i) is represented in the disk model, while the hypercycle in (ii) and horocycles in (iii) are represented in the half-plane model.

1.4 Remark (constant-curvature curves). Define an *Euclidean circle* to be either a line or a circle in \mathbb{C} ; its center is taken to be ∞ if it is a line.

- (a) Circles, hypercycles and horocycles are the orbits of elliptic, hyperbolic and parabolic one-parameter subgroups of isometries, respectively. In particular, they have constant curvature.
- (b) In the models D and H , a circle, horocycle or hypercycle appears as the intersection with \mathbb{H}^2 of an Euclidean circle which is disjoint, secant or (internally) tangent to $\partial\mathbb{H}^2$, respectively.
- (c) In the hyperboloid model, a circle, hypercycle or horocycle appears as a (planar) ellipse, hyperbola or parabola, respectively.
- (d) If all points of a circle lie at a distance $r > 0$ from a certain point, then its curvature is given by $\pm \coth r$. Thus, all circles have curvature greater than 1 in absolute value.
- (e) If all points of a hypercycle lie at a distance $r > 0$ from a certain geodesic, then its curvature is given by $\pm \tanh r$. Thus, all hypercycles have curvature less than 1 in absolute value.
- (f) The curvature of a horocycle equals ± 1 .
- (g) Circles, hypercycles, horocycles and their arcs account for all constant-curvature curves.

²The precise value of r is irrelevant, cf. (7.12).

Assertions (d), (e) and (f) may be proved by mapping the curve through an isometry to the corresponding curve in Figure 1 and applying Gauss-Bonnet to the shaded region. The remaining assertions are also straightforward consequences of the transitivity of the group of isometries on $UT\mathbb{H}^2$.

In view of the sign ambiguity in the preceding formulas, it is desirable to redefine circles, hypercycles and horocycles not as subsets of \mathbb{H}^2 , but as oriented curves therein. The *radius* $r \in \mathbb{R}$ of a circle or hypercycle is now defined so that its curvature κ is given by

$$\kappa = \coth r \quad \text{or} \quad \kappa = \tanh r,$$

respectively. Then $|r|$ is the distance from the circle (resp. hypercycle) to the point (resp. geodesic) to which it is equidistant. Both expressions for the curvature apply to horocycles if these are regarded as circles/hypercycles of radius $\pm\infty$, a convention which we shall adopt. Observe that in all cases the sign of r is the same as that of the curvature.

1.5 Remark (curvature and orientation). Let the circle at infinity be oriented from left to right in H and counter-clockwise in D . In either of these models (compare Figure 2):

- (a) If a hypercycle meets the circle at infinity at an angle $\alpha \in (0, \pi)$, then its curvature equals $\cos \alpha$. This follows from a reduction to the hypercycle depicted in Figure 1 (ii), with the indicated orientation, by expressing r in terms of α . More explicitly,

$$r = \int_{\alpha}^{\frac{\pi}{2}} \frac{1}{\sin t} dt = -\log \tan\left(\frac{\alpha}{2}\right) \quad (\alpha \in (0, \pi)),$$

so that $\tanh r = \cos \alpha$.

- (b) If a circle is oriented (counter-)clockwise, then its curvature is less than -1 (greater than 1). This follows immediately from a reduction to Figure 1 (i).
- (c) Both preceding assertions can be extended to include horocycles. This follows by representing horocycles as circles tangent to the circle at infinity in the disk model.

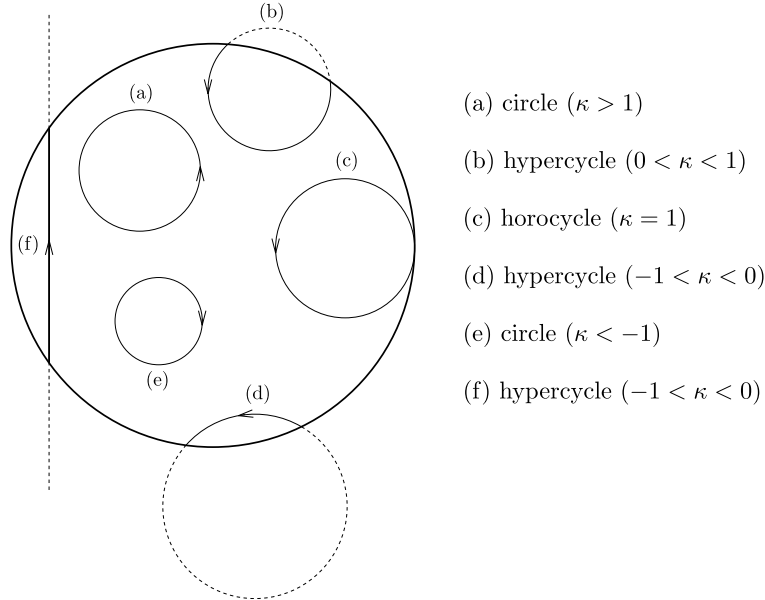


FIGURE 2. Examples of curves of constant curvature in the disk model. Note that the sign of the curvature of a hypercycle need not agree with that of its Euclidean curvature.

1.6 Exercise (families of parallel hypercycles). Let $p \neq q \in \mathbb{S}_{\infty}^1$ and $\kappa \in (-1, 1)$. Then there exist exactly two hypercycles of curvature κ through p and q . Furthermore, in the models D or H :³

- (a) This pair of hypercycles is related by inversion in \mathbb{S}_{∞}^1 , and their Euclidean centers lie along the Euclidean perpendicular bisector l of \overline{pq} .

³Exercises are not used anywhere else in the text and may be skipped without any loss.

Let o and o' denote the Euclidean centers of \mathbb{S}_∞^1 and the geodesic through the pair p, q , respectively.

- (b) Any point of l except for o and o' is the Euclidean center of a unique hypercycle of curvature in $(0, 1)$ through p and q . Describe its orientation in terms of the position of its center.
- (c) Using (1.5) (a), describe how its curvature changes as its center moves along l .

1.7 Definition (normal translation). Let $\gamma: [0, 1] \rightarrow \mathbb{H}^2$ be a regular curve and $\rho \in \mathbb{R}$. The *normal translation* $\gamma_\rho: [0, 1] \rightarrow \mathbb{H}^2$ of γ by ρ is the curve given by

$$\gamma_\rho(t) = \exp_{\gamma(t)}(\rho \mathbf{n}(t)) \quad (t \in [0, 1]),$$

where \mathbf{n} is the unit normal to γ and \exp the (Riemannian) exponential map. In the hyperboloid model,

$$(3) \quad \gamma_\rho(t) = \cosh \rho \gamma(t) + \sinh \rho \mathbf{n}(t) \quad (t \in [0, 1]).$$

Remark. The unit tangent bundle of any smooth surface has a canonical contact structure (cf. [25], §3.7). A curve $\tilde{\gamma}$ on UTS is Legendrian (with respect to this structure) if the velocity vector of its projection γ to S is always proportional to the vector prescribed by $\tilde{\gamma}$. When S is oriented, a (local) flow of contact automorphisms ϕ_ρ on UTS can be defined by letting $\phi_\rho(u)$ be the parallel translation of u along the geodesic perpendicular to u by a signed distance of ρ toward the left ($u \in UTS$, $\rho \in \mathbb{R}$). If S is complete, ϕ_ρ is defined on all of UTS for each $\rho \in \mathbb{R}$. The normal translation of γ as defined in (1.7) is nothing but the projection to S of $\phi_\rho(\mathbf{t}_\gamma)$, in the special case where $S = \mathbb{H}^2$. As will be seen below, this operation may create or remove cusps.

1.8 Remark (normal translation of constant-curvature curves in \mathbb{H}^2). Let $r, \rho \in \mathbb{R}$.

- (a) The normal translation by ρ of a circle of radius r is a circle of radius $r - \rho$, equidistant to the same point as the original circle.
- (b) The normal translation by ρ of a hypercycle of radius r is a hypercycle of radius $r - \rho$, equidistant to the same geodesic as the original hypercycle.
- (c) A normal translation of a horocycle is another horocycle, meeting orthogonally the same family of geodesics as the original horocycle.

More concisely, the normal translation of a constant-curvature curve of radius $r \in \mathbb{R} \cup \{\pm\infty\}$ by $\rho \in \mathbb{R}$ is a curve of the same type of radius $r - \rho$.

- (d) A normal translation of a hypercycle (resp. horocycle) meets $\partial\mathbb{H}^2$ in the same points (resp. point) as the original hypercycle (resp. horocycle), when represented in one of the models D or H .

Once again, to prove these assertions one can use an isometry to represent the circle (hypercycle, horocycle) as in Figure 1, where they become trivial.

Notice also that ρ goes from 0 to $2r$, the circle shrinks to a singularity ($\rho = r$) and then expands back to the original circle, but with reversed orientation ($\rho = 2r$). When $\rho = r$ the hypercycle becomes the geodesic, and when $\rho = 2r$ it becomes the other component of the locus of points at distance $|r|$ from the geodesic. This behavior is subsumed in the following result.

1.9 Lemma (normal translation of general curves). Let $\gamma: [0, 1] \rightarrow \mathbb{H}^2$ be smooth and regular,

$$\kappa_- = \min_{t \in [0, 1]} \kappa(t) \quad \text{and} \quad \kappa_+ = \max_{t \in [0, 1]} \kappa(t).$$

Assume that $\coth \rho \notin [\kappa_-, \kappa_+]$. Then:

- (a) The normal translation γ_ρ is regular. In particular, γ_ρ is regular for all ρ in some open interval containing 0, and for all $\rho \in \mathbb{R}$ in case $[\kappa_-, \kappa_+] \subset [-1, 1]$.
- (b) $\mathbf{t}_{\gamma_\rho} \equiv \mathbf{t}_\gamma$ if these are regarded as taking values in $\mathbb{E}^{2,1} \supset L$.

Given $t \in [0, 1]$, there exists a unique constant-curvature curve which osculates γ at $\gamma(t)$. The radius of curvature $r_\gamma(t)$ of γ at $\gamma(t)$ is defined as the radius of this osculating curve.

- (c) The radii of curvature of γ_ρ and γ are related by $r_{\gamma_\rho} = r_\gamma - \rho$.
- (d) The curvature of γ_ρ is given by:

$$\kappa_{\gamma_\rho}(t) = \begin{cases} \frac{1 - \kappa_\gamma(t) \coth \rho}{\kappa_\gamma(t) - \coth \rho} & \text{if } |\kappa_\gamma(t)| > 1; \\ \frac{\kappa_\gamma(t) - \tanh \rho}{1 - \kappa_\gamma(t) \tanh \rho} & \text{if } |\kappa_\gamma(t)| < 1; \\ \kappa_\gamma(t) & \text{if } |\kappa_\gamma(t)| = 1. \end{cases}$$

(e) $(\gamma_\rho)_{-\rho} = \gamma$.

Proof. For (a) and (b), use (3). It is clear from the definition of “osculation” that if η osculates γ at $\gamma(t)$, then η_ρ osculates γ_ρ at $\gamma_\rho(t)$. Thus part (c) is a consequence of (1.8). Part (d) follows from the addition formulas for \coth and \tanh , and part (e) is obvious. \square

The topology of $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ depends in principle upon eight real parameters: three for each of $u, v \in UT\mathbb{H}^2$, and two for the curvature bounds. This number can be halved by a suitable use of normal translations and isometries. In the sequel, two intervals are said to *overlap* if they intersect but neither is contained in the other one.

1.10 Proposition (parameter reduction). *Let $(\kappa_1, \kappa_2) \neq (-1, 1)$ and $u, v, \bar{u} \in UT\mathbb{H}^2$ be given. Then there exist $\bar{v} \in UT\mathbb{H}^2$ and κ_0 such that $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ is canonically homeomorphic to a space of the type listed in Table 1.*

Case	$\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ homeomorphic to	Range of κ_0
(κ_1, κ_2) contained in $[-1, 1]$	$\mathcal{C}_0^{\kappa_0}(\bar{u}, \bar{v})$	$(0, 1]$
(κ_1, κ_2) disjoint from $[-1, 1]$	$\mathcal{C}_{\kappa_0}^{+\infty}(\bar{u}, \bar{v})$	$[1, +\infty)$
(κ_1, κ_2) overlaps $[-1, 1]$	$\mathcal{C}_{\kappa_0}^{+\infty}(\bar{u}, \bar{v})$	$[-1, 1)$
(κ_1, κ_2) contains $[-1, 1]$	$\mathcal{C}_{-\kappa_0}^{+\kappa_0}(\bar{u}, \bar{v})$	$(1, +\infty]$

TABLE 1. Reduction of the number of parameters controlling the topology of $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$, for $(\kappa_1, \kappa_2) \neq (-1, 1)$.

The special role played by the interval $[-1, 1]$ stems from the fact that any normal translation of a horocycle is a horocycle. The four classes listed in Table 1 are genuinely different in terms of their topological properties, as discussed in the introduction. In the only case not covered by (1.10), namely $(\kappa_1, \kappa_2) = (-1, 1)$, given u, v, \bar{u} , one can find \bar{v} such that $\mathcal{C}_{-1}^{+1}(u, v)$ is homeomorphic to $\mathcal{C}_{-1}^{+1}(\bar{u}, \bar{v})$; for this it is sufficient to compose all curves in the former with an isometry taking u to \bar{u} .

Proof. The argument is similar for all four classes, so we consider in detail only the case where (κ_1, κ_2) overlaps $[-1, 1]$. Firstly, notice that composition with an orientation-reversing isometry switches the sign of the curvature of a curve. Therefore, by applying a reflection in some geodesic if necessary, it can be assumed that $-1 \leq \kappa_1 < 1 < \kappa_2$ (instead of $\kappa_1 < -1 < \kappa_2 \leq 1$). Write $\kappa_2 = \coth \rho_2$, $\kappa_1 = \tanh \rho_1$ and $\kappa_0 = \tanh(\rho_1 - \rho_2)$. Then normal translation by ρ_2 (i.e., the map $\gamma \mapsto \gamma_{\rho_2}$) provides a homeomorphism

$$\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v) \rightarrow \mathcal{C}_{\kappa_0}^{+\infty}(u', v'),$$

where u' is obtained by parallel translating u by a distance ρ_2 along the ray emanating from Ju , and similarly for v' . The inverse of this map is simply normal translation by $-\rho_2$. All details requiring verification here were already dealt with in (1.9). Finally, post-composition of curves with the unique orientation-preserving hyperbolic isometry taking u' to \bar{u} yields a homeomorphism

$$\mathcal{C}_{\kappa_0}^{+\infty}(u', v') \rightarrow \mathcal{C}_{\kappa_0}^{+\infty}(\bar{u}, \bar{v}).$$

In the remaining cases, let ρ_i denote the radius of a curve of constant curvature κ_i ($i = 1, 2$). For the first class in the table, apply a reflection if necessary to ensure that $\kappa_1 > -1$, then use normal translation by ρ_1 . For the second class, reduce to the case where $\kappa_1 \geq 1$ and apply normal translation by ρ_2 . For the fourth class, apply a normal translation by $\frac{1}{2}(\rho_1 + \rho_2)$. In all cases, use an orientation-preserving isometry to adjust the initial unit tangent vector to \bar{u} . \square

Remark. The homeomorphism constructed in the proof of (1.10) operates on the curves in a given space by a composition of a normal translation and an isometry. It is “canonical” in the sense that these two transformations, as well as the values of κ_0 and \bar{v} , are uniquely determined by κ_1, κ_2, u, v and \bar{u} . However, this does not preclude the existence of some more complicated homeomorphism between spaces in the second column of Table 1.

1.11 Exercise (extension of (1.10)). Let $(\kappa_1, \kappa_2) \neq (-1, 1)$ and $u, v, \bar{u} \in UT\mathbb{H}^2$ be given. Use the argument above to prove:

- (a) If (κ_1, κ_2) contains $[-1, 1]$, then there exist $\kappa_0 \in [-\infty, -1)$ and $\bar{v} \in UT\mathbb{H}^2$ such that

$$\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v) \approx \mathcal{C}_{\kappa_0}^{+\infty}(\bar{u}, \bar{v}).$$

- (b) If $-1 < \kappa_1 < \kappa_2 < 1$, then there exist $\kappa_0 \in (0, 1)$ and $\bar{v} \in UT\mathbb{H}^2$ such that

$$\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v) \approx \mathcal{C}_{-\kappa_0}^{+\kappa_0}(\bar{u}, \bar{v}).$$

Note that the hypothesis here is more restrictive than in the first case of the table.

2. VOIDNESS OF THE CANONICAL SUBSPACES

The purpose of this section is to discuss which of the canonical subspaces of $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ are empty. In the sequel $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, \cdot)$ (resp. $\bar{\mathcal{C}}_{\kappa_1}^{\kappa_2}(u, \cdot)$) denotes the space obtained from (1.1) by omitting the restriction that $\mathbf{t}_\gamma(1) = v$.

2.1 Lemma (attainable endpoints). *If at least one $|\kappa_i| > 1$, then $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v) \neq \emptyset$ for all $u, v \in UT\mathbb{H}^2$.*

Proof. By symmetry, we may assume that $\kappa_2 > 1$. Further, using normal translation by $\rho_2 = \operatorname{arccoth} \kappa_2$, we reduce to the case where $\kappa_2 = +\infty$. Let $q \in \mathbb{H}^2$ and $x \in UT\mathbb{H}_q^2$ be arbitrary. Consider the map

$$F_x: \mathcal{C}_{\kappa_1}^{+\infty}(x, \cdot) \rightarrow UT\mathbb{H}^2, \quad F_x(\gamma) = \mathbf{t}_\gamma(1).$$

It is not hard to see that this map is open; cf. (7.4). We begin by proving that $\operatorname{Im}(F_x) \supset UT\mathbb{H}_q^2$. Since sufficiently tight circles are allowed, $x \in \operatorname{Im}(F_x)$. Let $I \subset UT\mathbb{H}_q^2 \cap \operatorname{Im}(F_x)$ be an open interval about x . For each $y \in I$, let R_y denote the elliptic isometry fixing q and taking x to y . Let

$$\gamma_y \in \mathcal{C}_{\kappa_1}^{\kappa_2}(x, y) \quad \text{and} \quad \gamma_z \in \mathcal{C}_{\kappa_1}^{\kappa_2}(x, z) \quad (y, z \in UT\mathbb{H}_q^2).$$

Then the concatenation

$$\gamma_y * (R_y \circ \gamma_z)$$

starts at x and ends at $dR_y(z)$. This curve may not have a second derivative at the point of concatenation, but it may be approximated by a smooth curve in $\mathcal{C}_{\kappa_1}^{\kappa_2}(x, R_y z)$. This shows that $\operatorname{Im}(F_x)$ contains the interval $\{R_y z \mid y \in I\}$ about z whenever it contains z . In turn, this implies that $\operatorname{Im}(F_x)$ includes $UT\mathbb{H}_q^2$. Therefore, if $\operatorname{Im}(F_u)$ contains any tangent vector x based at q , it must also contain $UT\mathbb{H}_q^2 \subset \operatorname{Im}(F_x)$, by transitivity.

Now let $r := \operatorname{arccoth} \kappa_1$ in case $\kappa_1 > 1$ and $r := +\infty$ otherwise. If $\operatorname{Im}(F_u)$ contains tangent vectors at q , then it also contains tangent vectors at q' for any $q' \in B(q; 2r)$, since q' can be reached from q by the half-circle centered at the midpoint of the geodesic segment qq' of radius $\frac{1}{2}d(q, q') < r$. We conclude that $\operatorname{Im}(F_u) = UT\mathbb{H}^2$, that is, $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v) \neq \emptyset$ for all $v \in UT\mathbb{H}^2$. \square

The converse of (2.1) is an immediate consequence of the following result; cf. also (2.3).

2.2 Lemma (unattainable endpoints). *Let $(\kappa_1, \kappa_2) \subset [-1, 1]$ and $u \in UT\mathbb{H}^2$. The hypercycles (or horocycles) of curvature κ_1 and κ_2 tangent to u divide \mathbb{H}^2 into four open regions. Any curve in $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, \cdot)$ is confined to one of these regions R . Conversely, a point in R can be reached by a curve of constant curvature in $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, \cdot)$.*

Proof. The hypercycle of curvature $\frac{1}{2}(\kappa_1 + \kappa_2)$ tangent to u meets $\partial\mathbb{H}^2$ at two points; let R denote the open region whose closure contains the one point in $\partial\mathbb{H}^2$ towards which u is pointing. Let $\gamma \in \mathcal{C}_{\kappa_1}^{\kappa_2}(u, \cdot)$. Comparison of the curvatures of γ and those of the hypercycles bounding R shows that $\gamma(t) \in R$ for sufficiently small $t > 0$. Suppose for a contradiction that γ reaches ∂R for the first time at $t = T \in (0, 1]$ at some point of the hypercycle E of curvature $\kappa_2 = \tanh r$. (In case $\kappa_2 = 1$, E is a horocycle and $r = +\infty$.) The function

$$\delta: [0, T] \rightarrow \mathbb{R}, \quad \delta(t) = d(\gamma(t), E)$$

vanishes at 0 and T , hence it must attain its global maximum in $[0, T]$ at some $\tau \in (0, T)$, say $\delta(\tau) = \rho$. By (1.8), the normal translation of E by $-\rho$ is a hypercycle (resp. horocycle) of curvature $\tanh(r + \rho)$, which must be tangent to γ at $\gamma(\tau)$ because $\dot{\delta}(\tau) = 0$. Since δ has a local maximum at τ , comparison of curvatures yields that $\kappa_\gamma(\tau) \geq \tanh(r + \rho) > \kappa_2$, which is impossible.

The last assertion of the lemma holds because constant-curvature curves in $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, \cdot)$ foliate R . \square

Remark. With the notation of (2.2), the image of any curve in $\bar{\mathcal{C}}_{\kappa_1}^{\kappa_2}(u, \cdot)$ is contained in \bar{R} (see (1.1) for the definition of $\bar{\mathcal{C}}$). To prove this, let γ be a curve in this space. If $\kappa_\gamma \equiv \kappa_1$ or $\kappa_\gamma \equiv \kappa_2$, then the image of γ is entirely contained in ∂R . Otherwise, let $t_0 < 1$ be the infimum of all $t \in [0, 1]$ such that $\kappa_\gamma(t) \in (\kappa_1, \kappa_2)$, and apply the argument of (2.2) to $\gamma|_{[t_0, 1]}$.

2.3 Corollary. *The spaces $\mathcal{C}_{\kappa_1}^{\kappa_2}(\cdot, \cdot)$ and $\bar{\mathcal{C}}_{\kappa_1}^{\kappa_2}(\cdot, \cdot)$ contain closed curves if and only if at least one $|\kappa_i| > 1$.* \square

2.4 Corollary. *Let S be a hyperbolic surface. Then any closed curve whose curvature is bounded by 1 in absolute value is homotopically non-trivial (that is, it cannot represent the unit element of $\pi_1 S$).*

Proof. Indeed, the previous corollary shows that the lift of such a curve to \mathbb{H}^2 cannot be closed. \square

It will be convenient to introduce another model for \mathbb{H}^2 , which bears some similarity to Mercator world maps.

2.5 Definition (Mercator model). The underlying set of the *Mercator model* M is $(0, \pi) \times \mathbb{R}$. Its metric is defined by declaring the correspondence

$$M \rightarrow H, \quad (x, y) \mapsto e^{y+ix} \quad (x \in (0, \pi), y \in \mathbb{R})$$

to be an isometry. Because this map is the composition of the complex exponential with a reflection in the line $y = x$, M is conformal.

2.6 Remark (geometry of M). It is straightforward to verify that in the Mercator model M :

- (a) Vertical lines $y \mapsto (x, y)$, or parallels, represent hypercycles of curvature $\cos x$, corresponding in H to rays having $0 \in H$ for their initial point (see (1.5)(a)). Horizontal segments, or meridians, are geodesics corresponding in H to Euclidean half-circles centered at 0.
- (b) Vertical translations are hyperbolic isometries. The Riemannian metric g is given by

$$g_{(x,y)} = \frac{dx^2 + dy^2}{\sin^2 x} \quad ((x, y) \in M).$$

- (c) The Christoffel symbols are given by

$$\Gamma_{ij}^k(x, y) = \begin{cases} 0 & \text{if } i + j + k \text{ is odd;} \\ (-1)^{1+ij} \cot x & \text{if } i + j + k \text{ is even.} \end{cases}$$

2.7 Lemma. *Let $(\kappa_1, \kappa_2) \subset [-1, 1]$ and u be the vector $1 \in \mathbb{S}^1$ based at $(\frac{\pi}{2}, 0) \in M$. Then the image of any curve in $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, \cdot)$ or $\bar{\mathcal{C}}_{\kappa_1}^{\kappa_2}(u, \cdot)$ is the graph of a function of x when represented in M . Conversely, if $(\kappa_1, \kappa_2) \not\subset [-1, 1]$, then there exist curves in these spaces which are not graphs.*

Proof. Suppose that $\gamma \in \bar{\mathcal{C}}_{\kappa_1}^{\kappa_2}(u, \cdot)$ is not the graph of function of x when represented in M , i.e., it is tangent to a parallel P_1 at some time, say, $t = 1$. Let P_0 be the parallel $x = \frac{\pi}{2}$, which is orthogonal to γ at $t = 0$ by hypothesis; note that P_0 is a geodesic. Let L be the meridian through $\gamma(1)$, which is a geodesic orthogonal to both P_0 and P_1 . Let γ_1 be the concatenation of γ and its reflection in L (with reversed orientation). Let γ_2 be the concatenation of γ_1 and its reflection in P_0 (again with reversed orientation). Then γ_2 is a closed curve, hence $(\kappa_1, \kappa_2) \not\subset [-1, 1]$ by (2.3).

Conversely, if $(\kappa_1, \kappa_2) \not\subset [-1, 1]$, then $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, \cdot)$ contains circles, which are closed and hence not graphs. \square

2.8 Proposition (attainable turnings). *Consider the decomposition (1) of $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ into canonical subspaces.*

- (a) *If (κ_1, κ_2) contains $[-1, 1]$, then all of its canonical subspaces are nonempty.*
- (b) *If (κ_1, κ_2) is contained in $[-1, 1]$, then at most one canonical subspace is nonempty.*
- (c) *If (κ_1, κ_2) overlaps or is disjoint from $[-1, 1]$, then infinitely many of the canonical subspaces are nonempty, and infinitely many are empty.*

Proof. We split the proof into parts.

- (a) By (2.1), $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v) \neq \emptyset$. Since $(\kappa_1, \kappa_2) \supset [-1, 1]$, we may concatenate a curve in this space with circles of positive or negative curvature, traversed multiple times, to attain any desired total turning.
- (b) By (1.10), we may assume that u is the vector in the statement of (2.7). Then the assertion becomes an immediate consequence of the latter.
- (c) By (2.1), $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v) \neq \emptyset$. Because we may concatenate any curve in $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ with a circle (of positive curvature if $\kappa_1 > -1$ and of negative curvature if $\kappa_2 < 1$) traversed multiple times, $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v; \tau) \neq \emptyset$ for infinitely many values of τ . The remaining assertion, that $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v; \tau) = \emptyset$ for infinitely many values of τ , is a consequence of (2.10) (c) below. \square

2.9 Definition (α_{\pm}). Given a regular curve $\gamma: [0, 1] \rightarrow \mathbb{H}^2$, define maps

$$\alpha_{\pm}: [0, 1] \rightarrow \mathbb{S}_{\infty}^1$$

by letting $\alpha_{\pm}(t)$ be the point where the geodesic ray emanating from $\pm \mathbf{n}(t)$ meets \mathbb{S}_{∞}^1 .

2.10 Lemma. *Let $u, v \in UT\mathbb{H}^2$ be fixed.*

- (a) *Two curves $\gamma, \bar{\gamma} \in \mathcal{C}_{\infty}^{+\infty}(u, v)$ lie in the same component of this space if and only if the associated maps $\alpha_+, \bar{\alpha}_+: [0, 1] \rightarrow \mathbb{S}_{\infty}^1$ defined in (2.9) have the same total turning.*
- (b) *If $\kappa_{\gamma} > -1$ everywhere, then α_- is monotone. Similarly, if $\kappa_{\gamma} < +1$, then α_+ is monotone.*
- (c) *If $\kappa_1 \geq -1$, then there exists τ_0 such that $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v; \tau)$ is empty for all $\tau < \tau_0$.*

Proof. It is clear that if $\gamma, \bar{\gamma}$ lie in the same component, then $\alpha_+ \simeq \bar{\alpha}_+$ and $\alpha_- \simeq \bar{\alpha}_-$. Conversely, if $\gamma, \bar{\gamma}$ do not lie in the same component, then $\bar{\gamma}$ must be homotopic (through regular curves) to the concatenation of γ with a circle traversed n times, for some $n \neq 0$. This yields a homotopy between $\bar{\alpha}_+$ and α_+ concatenated with a map of degree n .

For part (b), it suffices to approximate γ by its osculating constant-curvature curve at each point. If such a curve is a circle, place its center at the origin in the disk model. If it is a hypercycle, regard it as an Euclidean ray in H .

Finally, (c) is a corollary of (a) and (b). \square

2.11 Exercise. Let $\gamma: [0, 1] \rightarrow \mathbb{H}^2$ be a regular curve and $A_{\pm}(\gamma) \subset \mathbb{S}_{\infty}^1$ denote the images of $\alpha_{\pm}: [0, 1] \rightarrow \mathbb{S}_{\infty}^1$.

- (a) Each of $A_{\pm}(\gamma)$ is a closed arc, which may be a singleton or all of \mathbb{S}_{∞}^1 .
- (b) Regard γ as a curve in H , and let ∞ be the unique point of \mathbb{S}_{∞}^1 not on the real line in this model. Then ∞ lies in the complement of $A_+(\gamma) \cup A_-(\gamma)$ if and only if \mathbf{t}_{γ} is never horizontal.
- (c) Let η be a horocycle of curvature 1, tangent to \mathbb{S}_{∞}^1 at z . Then $A_+(\eta) = \{z\}$ while $A_-(\eta) = \mathbb{S}_{\infty}^1 \setminus \{z\}$. What happens if the curvature of η is -1 ? (*Hint*: reduce to the case where $z = \infty$.)

3. FRAME AND LOGARITHMIC DERIVATIVE

The group $\text{Iso}_+(\mathbb{H}^2)$ of all orientation-preserving isometries of the hyperbolic plane acts simply transitively on $UT\mathbb{H}^2$. Therefore, an element g of this group is uniquely determined by where it maps a fixed unit tangent vector u_0 . This yields a correspondence between the two sets, viz., $g \leftrightarrow gu_0$. The *frame*

$$(4) \quad \Phi = \Phi_{\gamma}: [0, 1] \rightarrow \text{Iso}_+(\mathbb{H}^2)$$

of a regular curve $\gamma: [0, 1] \rightarrow \mathbb{H}^2$ is the image of \mathbf{t}_{γ} under this correspondence, and the *logarithmic derivative* $\Lambda = \Lambda_{\gamma}: [0, 1] \rightarrow \text{L}(\text{Iso}(\mathbb{H}^2))$ is its infinitesimal version. More precisely, Λ is the translation of $\dot{\Phi}$ to the Lie algebra $\text{L}(\text{Iso}(\mathbb{H}^2))$, defined by

$$(5) \quad \dot{\Phi}(t) = TL^{\Phi(t)}(\Lambda(t)) \quad (t \in [0, 1]).$$

Here $TL^{\Phi(t)}$ denotes the derivative (at the identity) of left multiplication by $\Phi(t)$. Although Φ depends on the choice of u_0 , Λ does not.

To be explicit, let $S = \text{diag}(-1, 1, 1)$,

$$\text{O}_{2,1} = \{Q \in \text{GL}_3(\mathbb{R}) \mid Q^t S Q = S\}$$

be the group of isometries of $\mathbb{E}^{2,1}$ and

$$\mathrm{SO}_{2,1}^+ = \{Q \in \mathrm{O}_{2,1} \mid \det(Q) = 1 \text{ and } Q(L) = L\}$$

be the connected component of the identity, which is isomorphic to $\mathrm{Iso}_+(\mathbb{H}^2)$. The corresponding Lie algebra is

$$\mathfrak{so}_{2,1} = \{X \in \mathfrak{gl}_3(\mathbb{R}) \mid X^t S + SX = 0\}.$$

In the hyperboloid model L , the identification between $UT\mathbb{H}^2$ and $\mathrm{Iso}_+(\mathbb{H}^2)$ takes the canonical form

$$(6) \quad UTL_p \ni u \leftrightarrow \begin{pmatrix} | & | & | \\ p & u & p \otimes u \\ | & | & | \end{pmatrix} \in \mathrm{SO}_{2,1}^+,$$

where \otimes denotes the Lorentzian vector product in $\mathbb{E}^{2,1}$. (Recall that $u \otimes v = S(u \times v)$, where \times denotes the usual vector product of vectors in \mathbb{R}^3 .) Note that this correspondence is also of the form $gu_0 \leftrightarrow g$ described above ($g \in \mathrm{SO}_{2,1}^+$), for $u_0 = e_1 = (0, 1, 0) \in UTL$.

3.1 Exercise (computations in L). Let $\gamma: [0, 1] \rightarrow L$ be smooth and regular.

(a) Denoting differentiation with respect to the given parameter (resp. arc-length) by \cdot (resp. \cdot'):

$$\mathbf{t}' = \kappa \mathbf{n} + \gamma, \quad \mathbf{n}' = -\kappa \mathbf{t} \quad \text{and} \quad \kappa = \mathbf{t}' \cdot \mathbf{n} = \frac{1}{\|\dot{\gamma}\|} \dot{\mathbf{t}} \cdot \mathbf{n} = \frac{1}{\|\dot{\gamma}\|^2} \ddot{\gamma} \cdot \mathbf{n} = \frac{1}{\|\dot{\gamma}\|^3} \det(\gamma, \dot{\gamma}, \ddot{\gamma}).$$

In these formulas $\gamma, \mathbf{t}, \mathbf{n}$ are viewed as taking values in $\mathbb{E}^{2,1}$, \cdot is the Lorentzian inner product and $\|\cdot\|^2$ the corresponding quadratic form.

(b) The frame $\Phi: [0, 1] \rightarrow \mathrm{SO}_{2,1}^+$ and logarithmic derivative $\Lambda: [0, 1] \rightarrow \mathfrak{so}_{2,1}$ of γ are given by

$$\Phi = \begin{pmatrix} | & | & | \\ \gamma & \mathbf{t} & \mathbf{n} \\ | & | & | \end{pmatrix} \quad \text{and} \quad \Lambda = \|\dot{\gamma}\| \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -\kappa \\ 0 & \kappa & 0 \end{pmatrix}.$$

Similar formulas for the curvature in the disk and half-plane models are much more cumbersome. However, the frame and logarithmic derivative do admit comparatively simple expressions.

For concreteness, we choose the correspondence between $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{Iso}_+(H)$ and UTH to be $M \leftrightarrow dM_i(1)$, where the latter denotes the image under the complex derivative dM of the tangent vector $1 \in \mathbb{C}$ based at $i \in H$. Similarly, we choose the correspondence between $\mathrm{Iso}_+(D)$ and UTD to be $M \leftrightarrow dM_0(\frac{1}{2})$. Recall that $\mathrm{Iso}_+(D)$ consists of those Möbius transformations of the form

$$z \mapsto \frac{az + b}{\bar{b}z + \bar{a}}, \quad |a|^2 - |b|^2 > 0 \quad (a, b \in \mathbb{C}).$$

3.2 Remark (Φ and Λ in the models D and H). Denote elements of projective groups as matrices in square brackets, the absolute value of a complex number by $|\cdot|$ and, exceptionally, the norm of a vector tangent to \mathbb{H}^2 by $\|\cdot\|$.

(a) Let $\gamma: [0, 1] \rightarrow H$ be a smooth regular curve. Then $\Lambda: [0, 1] \rightarrow \mathfrak{sl}_2(\mathbb{R})$ is given by

$$\Lambda = \frac{1}{2} \|\dot{\gamma}\| \begin{pmatrix} 0 & 1 + \kappa \\ 1 - \kappa & 0 \end{pmatrix}$$

and $\Phi: [0, 1] \rightarrow \mathrm{PSL}_2(\mathbb{R})$ is given by

$$\Phi = \begin{bmatrix} \mathrm{Im}(\gamma \bar{z}) & \mathrm{Re}(\gamma \bar{z}) \\ \mathrm{Im}(\bar{z}) & \mathrm{Re}(\bar{z}) \end{bmatrix}, \quad \text{where } \frac{\mathbf{t}}{|\mathbf{t}|} = z^2 \in \mathbb{S}^1.$$

(b) Let $\gamma: [0, 1] \rightarrow D$ be a smooth regular curve. Then $\Lambda: [0, 1] \rightarrow \mathfrak{sl}_2(\mathbb{C})$ is given by

$$\Lambda = \frac{1}{2} \|\dot{\gamma}\| \begin{pmatrix} i\kappa & 1 \\ 1 & -i\kappa \end{pmatrix}$$

and $\Phi: [0, 1] \rightarrow \mathrm{Iso}_+(D) \subset \mathrm{PSL}_2(\mathbb{C})$ is given by

$$\Phi = \begin{bmatrix} z & \gamma \bar{z} \\ \bar{\gamma} z & \bar{z} \end{bmatrix}, \quad \text{where } \frac{\mathbf{t}}{|\mathbf{t}|} = z^2 \in \mathbb{S}^1.$$

To establish the first formula, it suffices by (5) to consider the case the parameter is arc-length and $\Phi(s_0) = I \in \mathrm{PSL}_2(\mathbb{R})$. Without trying to find an expression for $\Phi(s)$ itself, write

$$M^s := \Phi(s) \in \mathrm{PSL}_2(\mathbb{R}), \quad M^s(z) = \frac{a(s)z + b(s)}{c(s)z + d(s)}, \quad M^s(i) = \gamma(s) \in H, \quad dM_i^s = \mathbf{t}(s) \in \mathbb{C}.$$

(Here dM^s denotes the complex derivative of M^s as a map $\mathbb{C} \rightarrow \mathbb{C}$.) Differentiate with respect to s at s_0 and express $\mathbf{t}'(s_0)$ in terms of $\frac{D\mathbf{t}}{ds}(s_0)$ to determine a', b', c', d' at s_0 . The derivation of the formula for Λ in (b) is analogous, and the expressions for Φ are obtained by straightforward computations.

A one-parameter group of hyperbolic isometries provides a foliation of \mathbb{H}^2 by its orbits, which are hypercycles by (1.4)(a). A regular curve $\gamma: [0, 1] \rightarrow \mathbb{H}^2$ admits *hyperbolic grafting* if there exist some such group G and $t_0, t_1 \in [0, 1]$ such that $\mathbf{t}(t_0)$ and $-\mathbf{t}(t_1)$ are tangent to orbits of G .

3.3 Remark (hyperbolic grafting). A regular curve $\gamma: [0, 1] \rightarrow H$ admits hyperbolic grafting if and only if there exist $t_0, t_1 \in [0, 1]$ such that $\Phi(t_1)\Phi(t_0)^{-1}$ has the form

$$\begin{bmatrix} r(1 - \sin 2\theta) & -r \cos 2\theta \\ \cos 2\theta & -(1 - \sin 2\theta) \end{bmatrix} = \begin{bmatrix} r(\cos \theta - \sin \theta) & -r(\cos \theta + \sin \theta) \\ \cos \theta + \sin \theta & \sin \theta - \cos \theta \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R})$$

for some $r > 0$ and $\theta \in (0, \frac{\pi}{2})$.

For the proof, note that any two one-parameter groups of hyperbolic isometries are conjugate, hence Γ may be taken as the group of positive homotheties centered at 0. By homogeneity, it may also be assumed that $\mathbf{t}(t_0) = i \in \mathbb{C}$ based at $i \in H$. Then $-\mathbf{t}(t_1) = \mathrm{Im}(z)z$ if it is based at $z \in H$ and tangent to an orbit of Γ . The Möbius transformations $M \in \mathrm{PSL}_2(\mathbb{R})$ satisfying $dM_i(i) = -\mathrm{Im}(z)z$ ($z = re^{2i\theta} \in H$) admit the stated description.

4. THE CASE WHERE (κ_1, κ_2) IS DISJOINT FROM $[-1, 1]$

4.1 Lemma. *In the disk and half-plane models, a curve whose (hyperbolic) curvature is everywhere greater than 1 is locally convex from the Euclidean viewpoint, i.e., its total turning is strictly increasing.*

Proof. It suffices to prove this for curves of constant curvature greater than 1, because a general curve satisfying the hypothesis is osculated by curves of this type. In D or H , such curves are represented as Euclidean circles traversed in the counterclockwise direction, hence they are locally convex. \square

4.2 Proposition. *If (κ_1, κ_2) is disjoint from $[-1, 1]$, then each canonical subspace of $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ is either empty or contractible.*

Proof. We will work in the half-plane model H throughout the proof. Points and tangent vectors will thus be regarded as elements of \mathbb{C} . By (1.10), it may be assumed that $\kappa_1 \geq 1$ and that u is parallel to $1 \in \mathbb{S}^1$. Let $\tau \in \mathbb{R}$ be a fixed valid total turning for curves in $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$, meaning that $e^{i\tau}$ is parallel to $v \in \mathbb{C}$.

Given $\gamma \in \mathcal{C}_{\kappa_1}^{\kappa_2}(u, v; \tau)$, let $\theta_\gamma: [0, 1] \rightarrow [0, \tau]$ be the unique continuous function such that $\theta_\gamma(0) = 0$ and

$$\frac{\mathbf{t}_\gamma(t)}{|\mathbf{t}_\gamma(t)|} = e^{i\theta_\gamma(t)} \text{ for all } t \in [0, 1],$$

where $|\cdot|$ denotes the usual absolute value of complex numbers. By (4.1), θ_γ is a diffeomorphism of $[0, 1]$ onto $[0, \tau]$. Thus it may be used as a parameter for γ .

Now fix $\gamma_0 \in \mathcal{C}_{\kappa_1}^{\kappa_2}(u, v; \tau)$ and set $\gamma_1 := \gamma$, both viewed as curves in H and parametrized by the argument $\theta \in [0, \tau]$, as above. Define

$$\gamma_s(\theta) = (1 - s)\gamma_0(\theta) + s\gamma_1(\theta) \in \mathbb{C} \quad (s \in [0, 1], \theta \in [0, \tau]).$$

Then each $\gamma_s: [0, \tau] \rightarrow H$ is a smooth curve satisfying $\mathbf{t}_{\gamma_s}(0) = u$ and $\mathbf{t}_{\gamma_s}(\tau) = v$. Moreover, it has the correct total turning τ , since $\mathbf{t}_{\gamma_s}(\theta)$ is always parallel to $e^{i\theta}$. By (1.4)(g) and the definition of “osculation”, the constant-curvature curves osculating γ from the Euclidean and hyperbolic viewpoints at any θ agree. Since $\kappa_1 \geq 1$, they are both equal to a certain circle completely contained in H , traversed counterclockwise; compare (1.5)(b). Therefore the osculating constant-curvature curve to γ_s at θ is another circle C_s , the corresponding convex combination of the osculating circles C_0 to γ_0 and

C_1 to γ_1 at θ (see Figure 3). Let y_s denote the y -coordinate of the Euclidean center of C_s and r_s its Euclidean radius ($s \in [0, 1]$). More explicitly,

$$y_s = (1-s)y_0 + sy_1 \quad \text{and} \quad r_s = (1-s)r_0 + sr_1.$$

The *hyperbolic* diameter d_s of C_s is given by

$$d_s = \log \left(\frac{y_s + r_s}{y_s - r_s} \right).$$

Note that $y_s > r_s$, as this is true for $s = 0, 1$. Because

$$2 \operatorname{arccoth}(\kappa_2) < d_i < 2 \operatorname{arccoth}(\kappa_1) \quad (i = 0, 1)$$

by hypothesis, a trivial computation shows that d_s satisfies the same inequalities for all $s \in [0, 1]$. Hence the curvature of γ_s does indeed take values inside (κ_1, κ_2) , and $(\gamma, s) \mapsto \gamma_s$ defines a contraction of $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v; \tau)$. \square

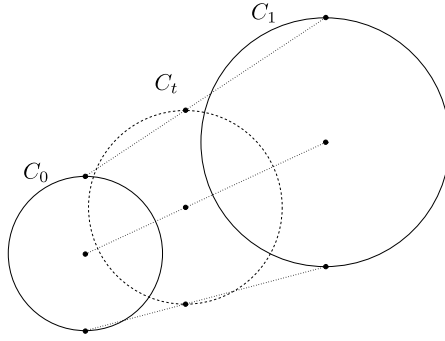


FIGURE 3. Proof of (4.2)

4.3 Corollary. *If (κ_1, κ_2) is disjoint from $[-1, 1]$, then $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ is homeomorphic to the disjoint union of countably many copies of the separable Hilbert space.*

Proof. This is an immediate consequence of (2.8)(c), (4.2) and (7.8)(b). \square

Remark. It is interesting to note that the proof of (4.2) does not work in the disk model. To understand what could go wrong, consider the situation where C_0 and C_1 have the same radius and the midpoint of the segment joining their centers is the origin of D (all concepts here being Euclidean). Then the hyperbolic radius of $C_{\frac{1}{2}}$ can be arbitrarily small compared to that of C_0 and C_1 .

Remark. The argument in the proof of (4.2) goes through without modifications to show that each component of $\bar{\mathcal{C}}_{\kappa_1}^{\kappa_2}(u, v)$ is contractible if $[\kappa_1, \kappa_2]$ is disjoint from $[-1, 1]$.

5. THE CASE WHERE $(\kappa_1, \kappa_2) \subset [-1, 1]$

In this section we will work with the spaces $\mathcal{L}_{\kappa_1}^{\kappa_2}(u, v)$ which are introduced in §7. These are larger than $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ in that they include regular piecewise C^2 curves. Our proof of (5.1) uses such curves and is therefore more natural in the former class. We recommend that the reader ignore the technical details for the moment and postpone a careful reading of §7. There is also a way to carry out the proof below in the setting of $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$: Whenever a discontinuity of the curvature arises, take an approximation by a smooth curve (which needs to be constructed); this path is more elementary but also cumbersome. The reader will notice that similar discussions appear in [17], [19] and [21].

Recall the Mercator model M defined in (2.5). Suppose that γ is a smooth regular curve in M which can be written in the form

$$(7) \quad \gamma: [a, b] \rightarrow M, \quad \gamma(x) = (x, y(x))$$

for some $[a, b] \subset (0, \pi)$; in other words, the image of γ is the graph of a function $y(x)$. A straightforward computation using (2.6) shows that the curvature of γ is then given by

$$(8) \quad \kappa_\gamma = \frac{1}{\sqrt{1 + \dot{y}^2}} \left(\frac{\ddot{y} \sin x}{1 + \dot{y}^2} - \dot{y} \cos x \right).$$

More important than this expression itself is the observation that it does not involve y , only its derivatives (because vertical translations are isometries of M). This can be exploited to express geometric properties of γ solely in terms of the function $f = \dot{y}$, and in particular to prove the following.⁴

5.1 Proposition. *If (κ_1, κ_2) is contained in $[-1, 1]$, then $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ is either empty or contractible, hence homeomorphic to the separable Hilbert space.*

Proof. By (1.10), we can assume that u is represented in M as the vector $1 \in \mathbb{S}^1$ based at $(\frac{\pi}{2}, 0)$. By (7.12), it suffices to prove that the Banach manifold $\mathcal{L}_{\kappa_1}^{\kappa_2}(u, v)$ is either empty or weakly contractible. Let

$$\mathbb{S}^k \rightarrow \mathcal{L}_{\kappa_1}^{\kappa_2}(u, v), \quad p \mapsto \gamma_p$$

be a continuous map. By (7.11), it can be assumed that each γ_p is smooth. In particular, it is possible to choose $\bar{\kappa}_2 < \kappa_2$ and $\bar{\kappa}_1 > \kappa_1$ such that the curvatures of all the γ_p take values inside $(\bar{\kappa}_1, \bar{\kappa}_2)$. By (2.7), any such curve γ may be parametrized as in (7). The function $f = \dot{y}$ satisfies the following conditions, whose meaning will be explained below:

- (i) $f(a) = \alpha$ and $f(b) = \beta$.
- (ii) $\int_a^b f(t) dt = A_0$.
- (iii) $\psi_{\bar{\kappa}_1}(x, f(x)) \leq \dot{f}(x) \leq \psi_{\bar{\kappa}_2}(x, f(x))$ for a.e. $x \in [a, b]$, where $\psi_\kappa: [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is given by:

$$\psi_\kappa(x, z) = \frac{1 + z^2}{\sin x} (z \cos x + \kappa \sqrt{1 + z^2}) \quad (\kappa \in [\bar{\kappa}_1, \bar{\kappa}_2], x \in [a, b], z \in \mathbb{R}).$$

In the present case, $a = \frac{\pi}{2}$ is the x -coordinate of $\gamma(a) = (\frac{\pi}{2}, 0)$ and b is the x -coordinate of $\gamma(b) \in M \subset \mathbb{C}$. The real numbers $\alpha = 0$ and β in (i) are the slopes of u and v regarded as vectors in \mathbb{C} . Condition (ii) prescribes the y -coordinate of $\gamma(b)$. Finally, the inequalities in (iii) express the fact that the curvature of γ takes values in $[\bar{\kappa}_1, \bar{\kappa}_2]$; compare (8). Conversely, suppose that an absolutely continuous function $f: [a, b] \rightarrow \mathbb{R}$ satisfies (i)–(iii). If we set

$$y(x) := \int_a^x f(t) dt$$

and define γ through (7), then $\gamma \in \mathcal{L}_{\kappa_1}^{\kappa_2}(u, v)$. Thus one can produce a contraction of the original family of curves by constructing a homotopy $(s, f) \mapsto f_s$ of the corresponding family of functions $f = f_1$, subject to the stated conditions throughout, with f_0 independent of f . This is what we shall now do.

For each $\kappa \in [\bar{\kappa}_1, \bar{\kappa}_2]$, let g_κ be the solution of the initial value problem

$$(9) \quad \dot{g}(x) = \psi_\kappa(x, g(x)), \quad g(a) = \alpha,$$

where α and ψ_κ are as in conditions (i) and (iii). Geometrically, the graph of g is an arc of the hypercycle of curvature κ with initial unit tangent vector u , represented in M . In particular, g_κ is defined over all of $[a, b]$ by (2.7). Similarly, for each $\kappa \in [\bar{\kappa}_1, \bar{\kappa}_2]$, let h_κ be the solution of the initial value problem

$$\dot{h}(x) = \psi_\kappa(x, h(x)), \quad h(b) = \beta.$$

Geometrically, the graph of h_κ is an arc of the hypercycle of curvature κ whose final unit tangent vector is v . Although h_κ is smooth, it is possible that it is not defined over all of $[a, b]$; if its maximal domain of definition is $(a', b]$ for some $a' > a$, then we extend h_κ to a function $[a, b] \rightarrow \mathbb{R} \cup \{\pm\infty\}$ by setting it equal to $\lim_{x \rightarrow a'^+} h_\kappa(x)$ over $[a, a']$.

Using this geometric interpretation, it follows from (2.2) that

$$g_{\bar{\kappa}_1}, h_{\bar{\kappa}_1} \leq f \leq g_{\bar{\kappa}_2}, h_{\bar{\kappa}_2}.$$

Moreover, from the same result one deduces that

$$(10) \quad g_\kappa(x) < g_{\kappa'}(x) \quad \text{for all } x > a \quad \text{if } \kappa < \kappa'.$$

⁴A similar construction was used in §3 of [21].

For each $\lambda, \mu \in [\bar{\kappa}_1, \bar{\kappa}_2]$, define $f_{(\lambda, \mu)}: [a, b] \rightarrow \mathbb{R}$ by

$$(11) \quad f_{(\lambda, \mu)}(x) = \text{median} \left(h_{\bar{\kappa}_1}(x), g_\lambda(x), f(x), g_\mu(x), h_{\bar{\kappa}_2}(x) \right).$$

Notice first that this function does not take on infinite values, since f, g_λ, g_μ are real functions. Similarly, since three of the functions above (namely, f, g_λ, g_μ) take the value α at a , and three of them (namely, $f, h_{\bar{\kappa}_1}, h_{\bar{\kappa}_2}$) take the value β at b , $f_{(\lambda, \mu)}$ automatically satisfies condition (i). It is easy to verify that it is Lipschitz and satisfies (iii) as well (see (5.2) below).

It remains to choose (λ, μ) appropriately to ensure that it satisfies condition (ii). Let

$$\begin{aligned} \kappa_+ &= \min \{ \kappa \in [\bar{\kappa}_1, \bar{\kappa}_2] \mid f(x) \leq g_\kappa(x) \text{ for all } x \in [a, b] \}, \\ \kappa_- &= \max \{ \kappa \in [\bar{\kappa}_1, \bar{\kappa}_2] \mid g_\kappa(x) \leq f(x) \text{ for all } x \in [a, b] \}, \\ \Delta &= \{ (\lambda, \mu) \in [\kappa_-, \kappa_+] \mid \lambda \leq \mu \} \quad (\text{cf. Figure 4}). \end{aligned}$$

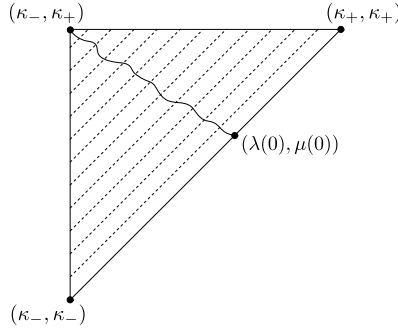


FIGURE 4. A diagram illustrating the triangle Δ in the (λ, μ) -plane. The dashed segments are the intersections of Δ with the lines $\{(\lambda, \mu) \in \mathbb{R}^2 \mid \mu - \lambda = s(\kappa_+ - \kappa_-)\}$ ($s \in [0, 1]$).

Define $A: \Delta \rightarrow \mathbb{R}$ to be the area under the graph of $f_{(\lambda, \mu)}$:

$$A(\lambda, \mu) = \int_a^b f_{(\lambda, \mu)}(x) dx.$$

Continuous dependence of the solutions of (9) on κ implies that A is a continuous function of (λ, μ) . Moreover, from

$$h_{\bar{\kappa}_1}, g_{\kappa_-} \leq f \leq g_{\kappa_+}, h_{\bar{\kappa}_2},$$

one deduces that

$$f_{(\kappa_-, \mu)} \leq f \leq f_{(\lambda, \kappa_+)}$$

for all $\lambda, \mu \in [\kappa_-, \kappa_+]$. Consequently, because the integral of f equals A_0 , for each $s \in [0, 1]$ the set

$$\{(\lambda, \mu) \in \Delta \mid \mu - \lambda = s(\kappa_+ - \kappa_-) \text{ and } A(\lambda, \mu) = A_0\}$$

is nonempty (see Figure 4). In fact, it consists of a unique point $(\lambda(s), \mu(s))$. To establish this, it suffices to show that $A(\lambda, \mu)$ is a strictly increasing function of both λ and μ . Now if

$$\kappa_- \leq \lambda < \lambda' \leq \mu \leq \kappa_+,$$

then the set of all $x \in [a, b]$ for which $g_\lambda(x) < f(x) < g_{\lambda'}(x)$ is nonempty, by (10) and the choice of κ_\pm . Therefore $f_{(\lambda, \mu)} < f_{(\lambda', \mu)}$ holds over a set of positive measure, while the nonstrict inequality holds everywhere by (10). This proves strict monotonicity with respect to λ ; the argument for μ is analogous. Continuity of A implies continuity of the curve $s \mapsto (\lambda(s), \mu(s)) \in \Delta$ (which is depicted in bold in Figure 4). The functions

$$f_s: [a, b] \rightarrow \mathbb{R}, \quad f_s = f_{(\lambda(s), \mu(s))}$$

satisfy all of conditions (i)–(iii) by construction, and they depend continuously on f and s . Let $\kappa_0 = \lambda(0) = \mu(0)$; then

$$f_0 = \text{median} (h_{\bar{\kappa}_1}, g_{\kappa_0}, h_{\bar{\kappa}_2}).$$

By (10), there is at most one value of $\kappa \in [\bar{\kappa}_1, \bar{\kappa}_2]$ for which the integral of $\text{median}(h_{\bar{\kappa}_1}, g_\kappa, h_{\bar{\kappa}_2})$ equals A_0 . This implies that κ_0 , and hence f_0 , is independent of f . Therefore $(f, s) \mapsto f_s$ is indeed a contraction. \square

The following fact was used without proof above.

5.2 Lemma. *The function $f_{(\lambda, \mu)}$ of (11) is Lipschitz and satisfies (iii).*

Proof. More generally, let ϕ be the median of $\phi_1, \dots, \phi_{2n+1}: [a, b] \rightarrow \mathbb{R}$. If each ϕ_k is c -Lipschitz, then ϕ is c -Lipschitz. Hence ϕ is absolutely continuous and its derivative exists a.e.. Furthermore, if ϕ and each ϕ_k are differentiable at x , then $\phi'(x) = \phi'_k(x)$ for some k such that $\phi(x) = \phi_k(x)$. In particular, if each ϕ_k satisfies the inequalities in (iii) (with ϕ_k in place of f), then so does ϕ .

The function $f_{(\lambda, \mu)}$ does not immediately conform to this situation because $h_{\bar{\kappa}_i}$ ($i = 1, 2$) may take on infinite values. This can be circumvented by subdividing $[a, b]$ into at most three subintervals (where none, one or both of $h_{\bar{\kappa}_i}$ are infinite) and applying the preceding remarks. \square

6. SPACES OF CURVES WITH CONSTRAINED CURVATURE OF CLASS C^r

In this section we consider spaces of curves with constrained curvature on an arbitrary surface, not necessarily hyperbolic nor orientable. We study their behavior under covering maps and show that they are always nonempty if S is compact; this should be contrasted with (2.1).

A *surface* is a smooth Riemannian 2-manifold. Given a regular curve $\gamma: [0, 1] \rightarrow S$, its *unit tangent* $\mathbf{t} = \mathbf{t}_\gamma$ is the lift of γ to the unit tangent bundle UTS of S :

$$\mathbf{t}: [0, 1] \rightarrow UTS, \quad \mathbf{t}(t) = \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}.$$

Now let an orientation of $TS_{\gamma(0)}$ be fixed. The *unit normal* to γ is the map $\mathbf{n} = \mathbf{n}_\gamma: [0, 1] \rightarrow UTS$ determined by the requirement that $(\mathbf{t}(t), \mathbf{n}(t))$ is an orthonormal basis of $TS_{\gamma(t)}$ whose parallel translation to $\gamma(0)$ (along the inverse of γ) is positively oriented, for each $t \in [0, 1]$. Assuming γ is twice differentiable, its *curvature* $\kappa = \kappa_\gamma$ is given by

$$(12) \quad \kappa := \frac{1}{|\dot{\gamma}|} \left\langle \frac{D\mathbf{t}}{dt}, \mathbf{n} \right\rangle = \frac{1}{|\dot{\gamma}|^2} \left\langle \frac{D\dot{\gamma}}{dt}, \mathbf{n} \right\rangle.$$

Here D denotes covariant differentiation (along γ).

Remark. If S is nonorientable, it is more common to define the (unsigned) curvature of $\gamma: [0, 1] \rightarrow S$ by

$$\kappa = \frac{1}{|\dot{\gamma}|} \left| \frac{D\mathbf{t}}{dt} \right|.$$

If S is orientable, the usual definition coincides with (12), but \mathbf{n} is defined by the condition that $(\mathbf{t}(t), \mathbf{n}(t))$ be positively oriented with respect to a specified orientation of S , rather than the parallel translation of an orientation of $TS_{\gamma(0)}$. These two definitions are equivalent, since an orientation of $TS_{\gamma(0)}$ determines an orientation of S if the latter is orientable. The definition that we have chosen has the advantage of allowing arbitrary bounds for the curvature of a curve on a nonorientable surface, and in particular the concise formulation of (6.3) below.

A geometric interpretation for the curvature is the following: Let $v: [0, 1] \rightarrow UTS$ be any smooth parallel vector field along γ , and let $\theta: [0, 1] \rightarrow \mathbb{R}$ be a function measuring the oriented angle from $v(t)$ to $\mathbf{t}(t)$. Then a trivial computation shows that $\dot{\theta} = \kappa |\dot{\gamma}|$. In particular, the *total curvature*

$$\int_0^1 \kappa(t) |\dot{\gamma}(t)| dt$$

of γ equals $\theta(1) - \theta(0)$.

In all that follows, the curvature bounds $\kappa_1 < \kappa_2$ are allowed to take values in $\mathbb{R} \cup \{\pm\infty\}$, S denotes a surface and $u, v \in UTS$. Moreover, it is assumed that an orientation of TS_p , where p is the basepoint of u , has been fixed.

6.1 Definition (spaces of C^r curves). Define $\mathcal{CS}_{\kappa_1}^{\kappa_2}(u, v)^r$ to be the set, endowed with the C^r topology (for some $r \geq 2$), of all C^r regular curves $\gamma: [0, 1] \rightarrow S$ such that:

- (i) $\mathbf{t}_\gamma(0) = u$ and $\mathbf{t}_\gamma(1) = v$;
- (ii) $\kappa_1 < \kappa_\gamma(t) < \kappa_2$ for each $t \in [0, 1]$.

Remark. Of course, whether S is orientable or not, the topological properties (or even the voidness) of $\mathcal{CS}_{\kappa_1}^{\kappa_2}(u, v)$ may be sensitive to the choice of orientation for TS_p . More precisely, if \bar{S} denotes the same surface S with the opposite orientation of TS_p , then $\mathcal{CS}_{\kappa_1}^{\kappa_2}(u, v) = \mathcal{CS}_{-\kappa_2}^{-\kappa_1}(u, v)$.

It will follow from (7.12) that r is irrelevant in the sense that different values yield spaces which are homeomorphic. Because of this, $\mathcal{CS}_{\kappa_1}^{\kappa_2}(u, v)^r$ is denoted simply by $\mathcal{CS}_{\kappa_1}^{\kappa_2}(u, v)$ in this section.

6.2 Lemma. *Define $\mathcal{CS}_{\kappa_1}^{\kappa_2}(u, \cdot)$ as in (6.1), except that no condition is imposed on $\mathbf{t}_\gamma(1)$, and similarly for $\mathcal{CS}_{\kappa_1}^{\kappa_2}(\cdot, v)$ and $\mathcal{CS}_{\kappa_1}^{\kappa_2}(\cdot, \cdot)$. Then:*

- (a) $\mathcal{CS}_{\kappa_1}^{\kappa_2}(u, \cdot)$ and $\mathcal{CS}_{\kappa_1}^{\kappa_2}(\cdot, v)$ are contractible.
- (b) $\mathcal{CS}_{\kappa_1}^{\kappa_2}(\cdot, \cdot)$ is homotopy equivalent to UTS .

Proof. By (7.8)(b), to prove that $\mathcal{CS}_{\kappa_1}^{\kappa_2}(u, \cdot)$ is contractible, it is actually sufficient to show that it is weakly contractible. Let

$$K \rightarrow \mathcal{CS}_{\kappa_1}^{\kappa_2}(u, \cdot), \quad p \mapsto \gamma^p,$$

be a continuous map, where K is a compact space. By a preliminary homotopy, each γ^p may be reparametrized with constant speed. Let $\lambda(p) = \text{length}(\gamma^p)$ and $0 < \lambda < \inf_{p \in K} \lambda(p)$. The curves can be shrunk through the homotopy

$$(s, p) \mapsto \gamma_s^p, \quad \gamma_s^p(t) := \gamma^p\left(\lambda(p)^{-1}[(1-s)\lambda + s\lambda(p)]t\right) \quad (s, t \in [0, 1], p \in K)$$

so that all γ_0^p have length λ , which can be chosen smaller than the injectivity radius of S at the basepoint of u . Then each γ_0^p is determined solely by its curvature, and conversely any function $\kappa: [0, 1] \rightarrow (\kappa_1, \kappa_2)$ of class C^{r-2} determines a unique curve of constant speed λ in S having u for its initial unit tangent vector. But the set of all such functions is convex.

Reversal of orientation of curves yields a homeomorphism between $\mathcal{CS}_{-\kappa_2}^{-\kappa_1}(-v, \cdot)$ and $\mathcal{CS}_{\kappa_1}^{\kappa_2}(\cdot, v)$, hence a space of the latter type is also contractible.

For (b), consider the map $f: UTS \rightarrow \mathcal{CS}_{\kappa_1}^{\kappa_2}(\cdot, \cdot)$ which associates to u the unique curve of constant curvature $\frac{1}{2}(\kappa_1 + \kappa_2)$ having u for its initial unit tangent and length equal to half the injectivity radius of S at the basepoint of u , parametrized with constant speed. Using the argument of the first paragraph, one deduces that f is a weak homotopy inverse of

$$g: \mathcal{CS}_{\kappa_1}^{\kappa_2}(\cdot, \cdot) \rightarrow UTS, \quad g(\gamma) = \mathbf{t}_\gamma(0).$$

Therefore $\mathcal{CS}_{\kappa_1}^{\kappa_2}(\cdot, \cdot)$ is homotopy equivalent to UTS , by (7.8)(b). \square

6.3 Lemma. *Let $q: \tilde{S} \rightarrow S$ be a Riemannian covering (a covering map which is also a local isometry) and $u, v \in UTS$. Suppose that $dq: (T\tilde{S}_{\tilde{p}}, \tilde{u}) \rightarrow (TS_p, u)$ preserves the chosen orientation of these tangent planes. Then $\tilde{\gamma} \mapsto q \circ \tilde{\gamma}$ yields a homeomorphism*

$$\bigsqcup_{\tilde{v} \in dq^{-1}(v)} \mathcal{CS}_{\kappa_1}^{\kappa_2}(\tilde{u}, \tilde{v}) \approx \mathcal{CS}_{\kappa_1}^{\kappa_2}(u, v).$$

Proof. Let $\gamma \in \mathcal{CS}_{\kappa_1}^{\kappa_2}(u, v)$ and $\tilde{\gamma}: [0, 1] \rightarrow \tilde{S}$ be its lift to \tilde{S} starting at \tilde{p} . Since dq is an isometry,

$$dq(\mathbf{t}_{\tilde{\gamma}}) = \mathbf{t}_\gamma \quad \text{and} \quad dq(\mathbf{n}_{\tilde{\gamma}}) = \pm \mathbf{n}_\gamma.$$

Moreover, $dq(\mathbf{n}_{\tilde{\gamma}}(0)) = \mathbf{n}_\gamma(0)$ by the hypothesis regarding orientations, hence $dq(\mathbf{n}_{\tilde{\gamma}}) = \mathbf{n}_\gamma$ by continuity. Now by (12), $\kappa_{\tilde{\gamma}} = \kappa_\gamma$. Therefore $\tilde{\gamma} \in \mathcal{CS}_{\kappa_1}^{\kappa_2}(\tilde{u}, \tilde{v})$ for some lift \tilde{v} of v . Conversely, if $\tilde{\gamma} \in \mathcal{CS}_{\kappa_1}^{\kappa_2}(\tilde{u}, \tilde{v})$, then $\gamma = q \circ \tilde{\gamma} \in \mathcal{CS}_{\kappa_1}^{\kappa_2}(u, v)$ by the same reasons. Since projection and lift (starting at \tilde{p}) are inverse operations, the asserted homeomorphism holds. \square

This result is especially useful when S is a space form (e.g., a hyperbolic surface); for then S is the quotient by a discrete group of isometries of the model simply-connected space of the same curvature, which is much more familiar. The lemma also furnishes a reduction to the orientable case by taking \tilde{S} to be the two-sheeted orientation covering of S .

6.4 Exercise. Suppose that (κ_1, κ_2) is symmetric about 0. Then the conclusion of (6.3) holds regardless of whether dq preserves orientation at \tilde{p} . (*Hint:* See the remark following (6.1).)

Remark. If closed or half-open intervals were used instead in (6.1), then substantial differences would only arise in marginal cases. For instance, in the situation of (2.2), if v is tangent to ∂R , then $\mathcal{C}_{\kappa_1}^{\kappa_2}(u, v)$ is empty, while $\bar{\mathcal{C}}_{\kappa_1}^{\kappa_2}(u, v)$ may not be. The original definition is more convenient to work with since the resulting spaces are Banach manifolds; compare Example 1.1 in [21].

6.5 Proposition. *Let S be a compact connected surface. Then $\mathcal{CS}_{\kappa_1}^{\kappa_2}(u, v) \neq \emptyset$ for any choice of $\kappa_1 < \kappa_2$ and $u, v \in UTS$.*

Proof. By passing to the orientation covering if necessary, it may be assumed that S is oriented. Let $\kappa_1 < \kappa_2$ be fixed. For $u, v \in UTS$, write $u \prec v$ if $\mathcal{CS}_{\kappa_1}^{\kappa_2}(u, v) \neq \emptyset$. Notice that \prec is transitive: Given curves in $\mathcal{CS}_{\kappa_1}^{\kappa_2}(u, v)$ and $\mathcal{CS}_{\kappa_1}^{\kappa_2}(v, w)$, their concatenation starts at u and ends at w ; its curvature may fail to exist at the point of concatenation, but this can be fixed by taking a smooth approximation. Let

$$F_u = \{v \in UTS \mid u \prec v\}, \quad E_u = \{v \in UTS \mid u \prec v \text{ and } v \prec u\}.$$

It is clear from the definition that $F_u \neq \emptyset$ for all u , and that F_u and E_u are open subsets of UTS (cf. (7.4)). Moreover, the family $(F_u)_{u \in UTS}$ covers UTS . Indeed, given v , we can find u such that $\mathcal{CS}_{-\kappa_2}^{-\kappa_1}(-v, -u) \neq \emptyset$. Reversing the orientation of a curve in the latter set, we establish that $\mathcal{CS}_{\kappa_1}^{\kappa_2}(u, v) \neq \emptyset$, that is, $v \in F_u$.

Since UTS is compact, it can be covered by finitely many of the F_u . Let $UTS = F_{u_1} \cup \dots \cup F_{u_m}$ be a minimal cover. We claim that $m = 1$.

Assume that $m > 1$. If $u_i \prec u_j$ then $F_{u_i} \supset F_{u_j}$, and therefore by minimality $i = j$. Since $u_i \in UTS$ and $u_i \notin F_{u_j}$ for $j \neq i$, we deduce that $u_i \in F_{u_i}$ for each i . The open sets E_{u_i} are thus nonempty and disjoint. On the other hand, every F_{u_i} must intersect some F_{u_j} with $j \neq i$, as UTS is connected. Choose $i \neq j$ such that $F_{u_i} \cap F_{u_j} \neq \emptyset$. It is easy to see that $F_{u_i} \cap F_{u_j}$ must be disjoint from E_{u_i} . Thus, if V is the interior of $F_{u_i} \setminus E_{u_i}$, then $V \neq \emptyset$. We will obtain a contradiction from this. By definition, there exist $u_* \in E_{u_i}$, $v_* \in V$ and $\gamma_* \in \mathcal{CS}_{\kappa_1}^{\kappa_2}(u_*, v_*)$. Let $\gamma_*: [0, L] \rightarrow S$ be parametrized by arc-length and $\kappa_*: [0, L] \rightarrow (\kappa_1, \kappa_2)$ denote its curvature.

The tangent bundle TS has a natural volume form coming from the Riemannian structure of S . A theorem of Liouville states that the geodesic flow preserves volume in TS . Since S is oriented, given $\theta \in \mathbb{R}$ we may define a volume-preserving bundle automorphism on UTS by $w \mapsto \cos \theta w + \sin \theta J(w)$ (where J is “multiplication by i ”). Let Y_0 and Z be the vector fields on UTS corresponding to the geodesic flow and to counterclockwise rotation, respectively. Then for any $\kappa \in \mathbb{R}$, the vector field $Y_\kappa = Y_0 + \kappa Z$ defines a volume-preserving flow on UTS ; the projections of its orbits on S are curves parametrized by arc-length of constant curvature κ .

By definition, the open set V is forward-invariant under each of these flows for $\kappa \in (\kappa_1, \kappa_2)$. Define a map $G: UTS \rightarrow UTS$ as follows: Given $u \in UTS$, $G(u) = \mathbf{t}_\eta(L)$, where $\eta: [0, L] \rightarrow S$ is the unique curve in $\mathcal{CS}_{\kappa_1}^{\kappa_2}(u, \cdot)$, parametrized by arc-length, whose curvature is κ_* . Then G must be volume-preserving, $G(V) \subset V$, but there exists a neighborhood of u_* contained in E_{u_i} which is taken by G to a neighborhood of v_* contained in V , a contradiction.

We conclude that $m = 1$, so that $UTS = F_{u_1}$. Furthermore, $F_{u_1} \setminus E_{u_1}$ must have empty interior by the preceding argument. Hence E_{u_1} is a dense open set in UTS . Let $u, v \in UTS$ be given. Since F_u is open, there exists $v_1 \in F_u \cap E_{u_1}$. Then $u \prec v_1$. Since $v_1 \prec u_1$ and $u_1 \prec v$, we deduce that $u \prec v$; in other words, $\mathcal{CS}_{\kappa_1}^{\kappa_2}(u, v) \neq \emptyset$. \square

Spaces of closed curves without basepoints. Just as in algebraic topology one considers homotopies with and without basepoint conditions, one can also study the space of all smooth closed curves on S with curvature in an interval (κ_1, κ_2) but no restrictions on the initial and final unit tangents. Let this space be denoted by $\mathcal{CS}_{\kappa_1}^{\kappa_2}$. In some regards this class may seem more fundamental than its basepointed version, the class of spaces $\mathcal{CS}_{\kappa_1}^{\kappa_2}(u, v)$ considered thus far. However, in the Hirsch-Smale theory of immersions, such basepoint conditions arise naturally. For instance, thm. C of [24] states that $\mathcal{CS}_{-\infty}^{+\infty}(u, u)$ is (weakly) homotopy equivalent to the loop space $\Omega(UTS)(u)$ consisting of all loops in UTS based at u . Moreover, even if one is interested only in closed curves, it is often helpful to study $\mathcal{CS}_{\kappa_1}^{\kappa_2}$ by lifting its elements to the universal cover of S , and these lifts need not be closed. A further

point is provided by the following result. Recall that UTS is diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^1$ if $S = \mathbb{R}^2$ or $S = \mathbb{H}^2$, and to $SO_3 \approx \mathbb{RP}^3$ if $S = \mathbb{S}^2$.

6.6 Lemma. *Let S be a simply-connected complete surface of constant curvature and $u \in UTS$ be arbitrary. Then $\mathcal{CS}_{\kappa_1}^{\kappa_2}$ is homeomorphic to $UTS \times \mathcal{CS}_{\kappa_1}^{\kappa_2}(u, u)$.*

Proof. The group of orientation-preserving isometries of such a surface acts simply transitively on UTS . Given $v \in UTS$, let g_v denote its unique element mapping v to u . Define

$$f: \mathcal{CS}_{\kappa_1}^{\kappa_2} \rightarrow UTS \times \mathcal{CS}_{\kappa_1}^{\kappa_2}(u, u), \quad f(\gamma) = (\mathbf{t}(0), g_{\mathbf{t}(0)} \circ \gamma).$$

This is clearly continuous, and so is its inverse, which is given by $(v, \eta) \mapsto g_v^{-1} \circ \eta$. \square

Regard the elements of $\mathcal{CS}_{\kappa_1}^{\kappa_2}$ as maps $\mathbb{S}^1 \rightarrow S$. There is a natural projection $\mathcal{CS}_{\kappa_1}^{\kappa_2} \rightarrow UTS$ taking a curve $\gamma: \mathbb{S}^1 \rightarrow S$ to its unit tangent at $1 \in \mathbb{S}^1$. This is a fiber bundle if $\kappa_1 = -\infty$ and $\kappa_2 = +\infty$ since S is locally diffeomorphic to \mathbb{R}^2 and the group of diffeomorphisms of the latter acts transitively on $UT\mathbb{R}^2$. It may be a fibration in certain other special cases, as in the situation of (6.6), but in general this cannot be guaranteed. For instance, if $S = T^2$ is a flat torus, then the homotopy type of the fibers $\mathcal{CS}_{-1}^{+1}(u, u)$ of the map $\mathcal{CS}_{-1}^{+1} \rightarrow UTS$ is not locally constant; this follows from an example in the introduction of [20]. We believe that little is known about the topology of $\mathcal{CS}_{\kappa_1}^{\kappa_2}$ beyond what is implied by (6.6).

7. SPACES OF CURVES WITH DISCONTINUOUS CURVATURE

Suppose that $\gamma: [0, 1] \rightarrow S$ is a smooth regular curve and, as always, $TS_{\gamma(0)}$ has been oriented. Let $\sigma: [0, 1] \rightarrow \mathbb{R}^+$ denote its speed $|\dot{\gamma}|$ and κ its curvature. Then γ and $\mathbf{t} = \mathbf{t}_\gamma: [0, 1] \rightarrow UTS$ satisfy:

$$(13) \quad \begin{cases} \dot{\gamma} = \sigma \mathbf{t} \\ \frac{D\mathbf{t}}{dt} = \sigma \kappa \mathbf{n} \end{cases} \quad \text{and} \quad \mathbf{t}(0) = u \in UTS.$$

Thus, γ is uniquely determined by u and the functions σ, κ . One can define a new class of spaces by relaxing the conditions that σ and κ be smooth.

Let $h: (0, +\infty) \rightarrow \mathbb{R}$ be the diffeomorphism

$$h(t) = t - t^{-1}.$$

For each pair $\kappa_1 < \kappa_2 \in \mathbb{R}$, let $h_{\kappa_1, \kappa_2}: (\kappa_1, \kappa_2) \rightarrow \mathbb{R}$ be the diffeomorphism

$$h_{\kappa_1, \kappa_2}(t) = (\kappa_1 - t)^{-1} + (\kappa_2 - t)^{-1}$$

and, similarly, set

$$\begin{aligned} h_{-\infty, +\infty}: \mathbb{R} &\rightarrow \mathbb{R}, & t &\mapsto t \\ h_{-\infty, \kappa_2}: (-\infty, \kappa_2) &\rightarrow \mathbb{R}, & t &\mapsto t + (\kappa_2 - t)^{-1} \\ h_{\kappa_1, +\infty}: (\kappa_1, +\infty) &\rightarrow \mathbb{R}, & t &\mapsto t + (\kappa_1 - t)^{-1}. \end{aligned}$$

Notice that all of these functions are monotone increasing, hence so are their inverses. Moreover, if $\hat{\kappa} \in L^2[0, 1]$, then $\kappa = h_{\kappa_1, \kappa_2}^{-1} \circ \hat{\kappa} \in L^2[0, 1]$ as well. This is obvious if (κ_1, κ_2) is bounded, and if one of κ_1, κ_2 is infinite, then it is a consequence of the fact that $h_{\kappa_1, \kappa_2}^{-1}(t)$ diverges linearly to $\pm\infty$ with respect to t . In what follows, \mathbb{L} denotes the separable Hilbert space $L^2[0, 1] \times L^2[0, 1]$.

7.1 Definition (admissible curve). A curve $\gamma: [0, 1] \rightarrow S$ is (κ_1, κ_2) -admissible if there exists $(\hat{\sigma}, \hat{\kappa}) \in \mathbb{L}$ such that γ satisfies (13) with

$$(14) \quad \sigma = h^{-1} \circ \hat{\sigma} \quad \text{and} \quad \kappa = h_{\kappa_1, \kappa_2}^{-1} \circ \hat{\kappa}.$$

When it is not important to keep track of the bounds κ_1, κ_2 , we will simply say that γ is *admissible*.

The system (13) has a unique solution for any $(\hat{\sigma}, \hat{\kappa}) \in \mathbb{L}$ and $u \in UTS$. To see this, we use coordinate charts for TS derived from charts for S and apply thm. C.3 on p. 386 of [26] to the resulting differential equation, noticing that S is smooth and $\sigma, \kappa \in L^2[0, 1] \subset L^1[0, 1]$. Furthermore, if we assume that S is complete, then the solution is defined over all of $[0, 1]$. The resulting maps $\gamma: [0, 1] \rightarrow S$ and

$\mathbf{t}: [0, 1] \rightarrow TS$ are absolutely continuous (see p. 385 of [26]), and so is \mathbf{n} . Using that $\langle \mathbf{t}, \mathbf{n} \rangle \equiv 0$ and differentiating, we obtain, in addition to (13), that

$$\frac{D\mathbf{n}}{dt} = -\sigma\kappa\mathbf{t} \quad \text{and} \quad |\mathbf{t}(t)| = |\mathbf{n}(t)| = |u| = 1 \quad \text{for all } t \in [0, 1].$$

Therefore, $\sigma = |\dot{\gamma}|$, $\mathbf{t}_\gamma = \mathbf{t}$ and $\mathbf{n}_\gamma = \mathbf{n}$. It is thus natural to call σ and κ the *speed* and *curvature* of γ , even though $\sigma, \kappa \in L^2[0, 1]$.

Remark. Although $\dot{\gamma} = \sigma\mathbf{t}$ is, in general, defined only almost everywhere on $[0, 1]$, if we reparametrize γ by arc-length then it becomes a regular curve, because $\gamma' = \mathbf{t}$ is continuous. It is helpful to regard admissible curves simply as regular curves whose curvatures are defined a.e..

7.2 Definition. For $u \in UTS$, let $\mathcal{L}S_{\kappa_1}^{\kappa_2}(u, \cdot)$ be the set of all (κ_1, κ_2) -admissible curves $\gamma: [0, 1] \rightarrow S$ with $\mathbf{t}_\gamma(0) = u$.

If S is complete, then this set is identified with \mathbb{L} via the correspondence $\gamma \leftrightarrow (\hat{\sigma}, \hat{\kappa})$, thus furnishing $\mathcal{L}S_{\kappa_1}^{\kappa_2}(u, \cdot)$ with a trivial Hilbert manifold structure. If S is not complete, then $\mathcal{L}S_{\kappa_1}^{\kappa_2}(u, \cdot)$ is some mysterious open subset of \mathbb{L} . However, we still have the following.

7.3 Lemma. For all $u \in UTS$, $\mathcal{L}S_{\kappa_1}^{\kappa_2}(u, \cdot)$ is homeomorphic to \mathbb{L} .

Proof. The proof is almost identical to that of (6.2)(a). Given a family of curves indexed by a compact space, first reparametrize all curves by constant speed and shrink them to a common length λ smaller than the injectivity radius of S at the basepoint of u . Now each curve is completely determined by its curvature, and conversely any L^2 -function $\hat{\kappa}: [0, 1] \rightarrow \mathbb{R}$ determines a unique curve of constant speed λ having u for its initial unit tangent vector, via (14). But the set of all such functions is convex. Thus $\mathcal{L}S_{\kappa_1}^{\kappa_2}(u, \cdot)$ is weakly contractible, hence homeomorphic to \mathbb{L} by (7.8)(a). \square

7.4 Lemma. Let S be a surface and define $F: \mathcal{L}S_{\kappa_1}^{\kappa_2}(u, \cdot) \rightarrow UTS$ by $\gamma \mapsto \mathbf{t}_\gamma(1)$. Then F is a submersion, and consequently an open map.

Proof. The proof when $S = \mathbb{R}^2$ is given in [21], Lemma 1.5. The proof in the general case follows by considering Riemannian normal coordinates, which are flat at least to second order, in a neighborhood of the basepoint of $\mathbf{t}_\gamma(1)$. \square

7.5 Definition (spaces of admissible curves). Define $\mathcal{L}S_{\kappa_1}^{\kappa_2}(u, v)$ to be the subspace of $\mathcal{L}S_{\kappa_1}^{\kappa_2}(u, \cdot)$ consisting of all γ such that $\mathbf{t}_\gamma(1) = v$.

It follows from (7.4) that $\mathcal{L}S_{\kappa_1}^{\kappa_2}(u, v)$ is a closed submanifold of codimension 3 in $\mathcal{L}S_{\kappa_1}^{\kappa_2}(u, \cdot) \approx \mathbb{L}$, provided it is not empty. We have already seen in (2.3) that such spaces may indeed be empty, but this cannot occur when S is compact; cf. (6.5) and (7.10).

Relations between spaces of curves. If $(\kappa_1, \kappa_2) \subset (\bar{\kappa}_1, \bar{\kappa}_2)$ and $\gamma \in \mathcal{L}S_{\kappa_1}^{\kappa_2}(u, v)$, then we can also consider γ as a curve in $\mathcal{L}S_{\bar{\kappa}_1}^{\bar{\kappa}_2}(u, v)$. However, the topology of the former space is strictly finer (i.e., has more open sets) than the topology induced by the resulting inclusion.

7.6 Lemma. Let $(\kappa_1, \kappa_2) \subset (\bar{\kappa}_1, \bar{\kappa}_2)$, S be a surface and $u \in UTS$. Then

$$(15) \quad (\hat{\sigma}, \hat{\kappa}) \mapsto (\hat{\sigma}, h_{\bar{\kappa}_1, \bar{\kappa}_2}^{-1} \circ h_{\kappa_1, \kappa_2}^{-1} \circ \hat{\kappa})$$

defines a continuous injection $j: \mathcal{L}S_{\kappa_1}^{\kappa_2}(u, \cdot) \rightarrow \mathcal{L}S_{\bar{\kappa}_1}^{\bar{\kappa}_2}(u, \cdot)$. The actual curves on S corresponding to these pairs are the same, but j is not a topological embedding unless $\bar{\kappa}_1 = \kappa_1$ and $\bar{\kappa}_2 = \kappa_2$.

Proof. It may be assumed that S is oriented. The curve γ corresponding to $(\hat{\sigma}, \hat{\kappa}) \in \mathcal{L}S_{\kappa_1}^{\kappa_2}(u, v)$ is obtained as the solution of (13) with

$$\sigma = h^{-1} \circ \hat{\sigma} \quad \text{and} \quad \kappa = h_{\kappa_1, \kappa_2}^{-1} \circ \hat{\kappa}.$$

The curve η corresponding to the right side of (15) in $\mathcal{L}S_{\bar{\kappa}_1}^{\bar{\kappa}_2}(u, \cdot)$ is the solution of (13) with

$$\sigma = h^{-1} \circ \hat{\sigma} \quad \text{and} \quad \kappa = h_{\bar{\kappa}_1, \bar{\kappa}_2}^{-1} \circ (h_{\bar{\kappa}_1, \bar{\kappa}_2}^{-1} \circ h_{\kappa_1, \kappa_2}^{-1} \circ \hat{\kappa}) = h_{\kappa_1, \kappa_2}^{-1} \circ \hat{\kappa}.$$

By uniqueness of solutions, $\gamma = \eta$. In particular, j is injective.

Set $g = h_{\bar{\kappa}_1, \bar{\kappa}_2} \circ h_{\kappa_1, \kappa_2}^{-1}$. Observe that

$$\lim_{t \rightarrow +\infty} g'(t) = \begin{cases} 1 & \text{if } \bar{\kappa}_2 = \kappa_2; \\ 0 & \text{otherwise;} \end{cases} \quad \lim_{t \rightarrow -\infty} g'(t) = \begin{cases} 1 & \text{if } \bar{\kappa}_1 = \kappa_1; \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $|g'|$ is bounded over \mathbb{R} . Consequently, there exists $C > 0$ such that

$$\|g \circ f_1 - g \circ f_2\|_2 \leq C \|f_1 - f_2\|_2 \quad \text{for any } f_1, f_2 \in L^2[0, 1].$$

We conclude that $j: (\hat{\sigma}, \hat{\kappa}) \mapsto (\hat{\sigma}, g \circ \hat{\kappa})$ is continuous.

Suppose now that $(\kappa_1, \kappa_2) \subsetneq (\bar{\kappa}_1, \bar{\kappa}_2)$. No generality is lost in assuming that $\kappa_2 < \bar{\kappa}_2$. Let

$$m = g(0) \quad \text{and} \quad M = g(+\infty) = h_{\bar{\kappa}_1, \bar{\kappa}_2}(\kappa_2).$$

Define a sequence of L^2 functions $\hat{\kappa}_n: [0, 1] \rightarrow \mathbb{R}$ by:

$$\hat{\kappa}_n(t) = \begin{cases} n & \text{if } t \in [\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}]; \\ 0 & \text{otherwise.} \end{cases} \quad (n \in \mathbb{N}^+, t \in [0, 1]).$$

Since g is the composite of increasing functions, $g(t) < g(+\infty) = M$ for any $t \in \mathbb{R}$. Therefore,

$$|g \circ \hat{\kappa}_n(t) - m| \begin{cases} \leq M - m & \text{if } t \in [\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}]; \\ = 0 & \text{otherwise.} \end{cases}$$

Hence, $\|\hat{\kappa}_n\|_2 = \sqrt{n} \rightarrow +\infty$ as n increases, while $\|g \circ \hat{\kappa}_n - m\|_2 \leq n^{-\frac{1}{2}}(M - m) \rightarrow 0$. We conclude that j is not a topological embedding. This argument may be modified to prove that $\mathcal{L}S_{\kappa_1}^{\kappa_2}(u, v) \hookrightarrow \mathcal{L}S_{\bar{\kappa}_1}^{\bar{\kappa}_2}(u, v)$ is likewise not an embedding for any v . \square

7.7 Lemma. *Let $(\kappa_1, \kappa_2) \subset (\bar{\kappa}_1, \bar{\kappa}_2)$ and $u, v \in UTS$. Then*

$$(16) \quad j: \mathcal{CS}_{\kappa_1}^{\kappa_2}(u, v)^r \rightarrow \mathcal{LS}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(u, v), \quad \gamma \mapsto (h \circ |\dot{\gamma}|, h_{\bar{\kappa}_1}^{\bar{\kappa}_2} \circ \kappa_\gamma)$$

is a continuous injection, but not an embedding, for all $r \geq 2$. Moreover, the actual curve on S corresponding to $j(\gamma)$ is γ itself.

Proof. The proof is very similar to that of (7.6). \square

The following lemmas contain all the results on infinite-dimensional manifolds that we shall need.⁵

7.8 Lemma. *Let \mathcal{M}, \mathcal{N} be (infinite-dimensional) Banach manifolds. Then:*

- (a) *If \mathcal{M}, \mathcal{N} are weakly homotopy equivalent, then they are in fact homeomorphic (diffeomorphic if \mathcal{M}, \mathcal{N} are Hilbert manifolds).*
- (b) *If the Banach manifold \mathcal{M} and the finite-dimensional manifold M are weakly homotopy equivalent, then \mathcal{M} is homeomorphic to $M \times \mathbb{L}$; in particular, \mathcal{M} and M are homotopy equivalent.*
- (c) *Let E and F be separable Banach spaces. Suppose $i: F \rightarrow E$ is a bounded, injective linear map with dense image and $\mathcal{M} \subset E$ is a smooth closed submanifold of finite codimension. Then $\mathcal{N} = i^{-1}(\mathcal{M})$ is a smooth closed submanifold of F and $i: \mathcal{N} \rightarrow \mathcal{M}$ is a homotopy equivalence.*

Proof. Part (a) follows from thm. 15 in [15] and cor. 3 in [9]. For part (b), apply (a) to \mathcal{M} and $\mathcal{N} = M \times \mathbb{L}$. Part (c) is thm. 2 in [4]. \square

7.9 Lemma. *Let \mathbb{L} be a separable Hilbert space, $D \subset \mathbb{L}$ a dense vector subspace, $L \subset \mathbb{L}$ a submanifold of finite codimension and U an open subset of L . Then the set inclusion $D \cap U \rightarrow U$ is a weak homotopy equivalence.*

Proof. We shall prove the lemma when $L = h^{-1}(0)$ for some submersion $h: V \rightarrow \mathbb{R}^n$, where V is an open subset of \mathbb{L} . This is sufficient for our purposes and the general assertion can be deduced from this by using a partition of unity subordinate to a suitable cover of L .

Let T be a tubular neighborhood of U in V such that $T \cap L = U$. Let K be a compact simplicial complex and $f: K \rightarrow U$ a continuous map. We shall obtain a continuous $H: [0, 2] \times K \rightarrow U$ such that $H(0, a) = f(a)$ for every $a \in K$ and $H(\{2\} \times K) \subset D \cap U$. Let e_j denote the j -th vector in

⁵(7.8) and a weaker version of (7.9) have already appeared in [19].

the canonical basis for \mathbb{R}^n , $e_0 = -\sum_{j=1}^n e_j$ and let $\Delta \subset \mathbb{R}^n$ denote the n -simplex $[e_0, \dots, e_n]$. Let $[x_0, x_1, \dots, x_n] \subset T$ be an n -simplex and $\varphi: \Delta \rightarrow [x_0, x_1, \dots, x_n]$ be given by

$$\varphi\left(\sum_{j=0}^n s_j e_j\right) = \sum_{j=0}^n s_j x_j, \quad \text{where } \sum_{j=0}^n s_j = 1 \quad \text{and } s_j \geq 0 \quad \text{for all } j = 0, \dots, n.$$

We shall say that $[x_0, x_1, \dots, x_n]$ is *neat* if $h \circ \varphi: \Delta \rightarrow \mathbb{R}^n$ is an embedding and $0 \in (h \circ \varphi)(\text{Int } \Delta)$.

Given $p \in T$, let dh_p denote the derivative of h at p and $N_p = \ker(dh_p)$. Define $w_j: T \rightarrow \mathbb{L}$ by:

$$(17) \quad w_j(p) = (dh_p|_{N_p^\perp})^{-1}(e_j) \quad (p \in T, j = 0, \dots, n).$$

Notice that $h(p + \sum_j \lambda_j w_j(p)) = h(p) + \sum_j \lambda_j e_j + o(|\lambda|)$ (for $\lambda = (\lambda_0, \dots, \lambda_n)$ and $p \in T$). Hence, using compactness of K , we can find $r, \varepsilon > 0$ such that:

- (i) For any $p \in f(K)$, $[p + rw_0(p), \dots, p + rw_n(p)] \subset T$ and it is neat;
- (ii) If $p \in f(K)$ and $|q_j - (p + rw_j(p))| < \varepsilon$ for each j , then $[q_0, \dots, q_n] \subset T$ and it is neat.

Let a_i ($i = 1, \dots, m$) be the vertices of the triangulation of K . Set $v_i = f(a_i)$ and

$$v_{ij} = v_i + rw_j(v_i) \quad (i = 1, \dots, m, j = 0, \dots, n).$$

For each such i, j , choose $\tilde{v}_{ij} \in D \cap T$ with $|\tilde{v}_{ij} - v_{ij}| < \frac{\varepsilon}{2}$. Let

$$(18) \quad \begin{aligned} v_{ij}(s) &= (2-s)v_{ij} + (s-1)\tilde{v}_{ij}, \quad \text{so that} \\ |v_{ij}(s) - v_{ij}| &< \frac{\varepsilon}{2} \quad (s \in [1, 2], i = 1, \dots, m, j = 0, \dots, n). \end{aligned}$$

For any $i, i' \in \{1, \dots, m\}$ and $j = 0, \dots, n$, we have

$$|v_{ij} - v_{i'j}| \leq |f(a_i) - f(a_{i'})| + r|w_j \circ f(a_i) - w_j \circ f(a_{i'})|.$$

Since f and the w_j are continuous functions, we can suppose that the triangulation of K is so fine that $|v_{ij} - v_{i'j}| < \frac{\varepsilon}{2}$ for each $j = 0, \dots, n$ whenever there exists a simplex having $a_i, a_{i'}$ as two of its vertices. Let $a \in K$ lie in some d -simplex of this triangulation, say, $a = \sum_{i=1}^{d+1} t_i a_i$ (where each $t_i > 0$ and $\sum_i t_i = 1$). Set

$$z_j(s) = \sum_{i=1}^{d+1} t_i v_{ij}(s) \quad (s \in [1, 2], j = 0, \dots, n).$$

Then $[z_0(s), \dots, z_n(s)]$ is a neat simplex because condition (ii) is satisfied (with $p = v_1$):

$$\left| \sum_{i=1}^{d+1} t_i v_{ij}(s) - v_{1j} \right| \leq \sum_{i=1}^{d+1} t_i (|v_{ij}(s) - v_{ij}| + |v_{ij} - v_{1j}|) < \varepsilon,$$

the strict inequality coming from (18) and our hypothesis on the triangulation. Define $H(s, a)$ as the unique element of $h^{-1}(0) \cap [z_0(s), \dots, z_n(s)]$ ($s \in [1, 2]$). Observe that for any $a \in K$, $H(s, a) \in U = h^{-1}(0) \cap T$ ($s \in [1, 2]$) and $H(2, a) \in D \cap U$, as it is the convex combination of the $\tilde{v}_{ij} \in D$.

By reducing $r, \varepsilon > 0$ (and refining the triangulation of K) if necessary, we can ensure that

$$(1-s)f(a) + sH(1, a) \in T \quad \text{for all } s \in [0, 1] \text{ and } a \in K.$$

Let $\text{pr}: T \rightarrow U$ be the associated retraction. Complete the definition of H by setting:

$$H(s, a) = \text{pr}((1-s)f(a) + sH(1, a)) \quad (s \in [0, 1], a \in K).$$

The existence of H shows that f is homotopic within U to a map whose image is contained in $D \cap U$. Taking $K = \mathbb{S}^k$, we conclude that the set inclusion $D \cap U \rightarrow U$ induces surjective maps $\pi_k(D \cap U) \rightarrow \pi_k(U)$ for all $k \in \mathbb{N}$.

We now establish that the inclusion $D \cap U \rightarrow U$ induces injections on all homotopy groups. Let $k \in \mathbb{N}$, $G: \mathbb{D}^{k+1} \rightarrow U$ be continuous and suppose that the image of $g = G|_{\mathbb{S}^k}$ is contained in $D \cap U$. Let $G_0: \mathbb{D}^{k+1} \rightarrow D \cap U$ be a close approximation to G ; the existence of G_0 was proved above. Let $\varepsilon \in (0, 1)$ and define

$$G_1: \mathbb{D}^{k+1} \rightarrow D \cap T \quad \text{by} \quad G_1(a) = \begin{cases} (1-s)g\left(\frac{a}{|a|}\right) + sG_0\left(\frac{a}{|a|}\right) & \text{if } |a| = (1-s\varepsilon), s \in [0, 1] \\ G_0\left(\frac{a}{1-\varepsilon}\right) & \text{if } |a| \leq 1-\varepsilon \end{cases}$$

Notice that we can make G_1 as close as desired to G by a suitable choice of G_0 and ε . Let w_j be as in (17). We claim that there exist continuous functions $\tilde{w}_j: \mathbb{D}^{k+1} \rightarrow D$ ($j = 0, \dots, n$) such that:

- (iii) $\sum_{j=0}^n \tilde{w}_j(a) = 0$ for all $a \in \mathbb{D}^{k+1}$;
- (iv) For any $a \in \mathbb{D}^{k+1}$, $[G_1(a) + \tilde{w}_0(a), \dots, G_1(a) + \tilde{w}_n(a)] \subset D \cap T$ and it is neat.

To prove this, invoke condition (ii) above (with \mathbb{D}^{k+1} in place of K and G in place of f) together with denseness of D to find constant \tilde{w}_j on open sets which cover \mathbb{D}^{k+1} , and use a partition of unity. By (iv), for each $a \in \mathbb{D}^{k+1}$ there exist unique $t_0(a), \dots, t_n(a) \in [0, 1]$ such that $\sum_i t_i(a) = 1$ and

$$G_2(a) = G_1(a) + t_0(a)\tilde{w}_0(a) + \dots + t_n(a)\tilde{w}_n(a) \in h^{-1}(0).$$

We obtain thus a continuous map $G_2: \mathbb{D}^{k+1} \rightarrow D \cap U$. Since $G_1|_{\mathbb{S}^k} = g$ and $h \circ g = 0$, we conclude from (iii) and uniqueness of the t_i that $G_2|_{\mathbb{S}^k} = g$. Therefore, G_2 is a nullhomotopy of g in $D \cap U$. \square

7.10 Corollary. *The subset of all smooth curves in $\mathcal{L}S_{\kappa_1}^{\kappa_2}(u, v)$ is dense in the latter.*

Proof. Take $\mathbb{L} = L^2[0, 1] \times L^2[0, 1]$, $D = C^\infty[0, 1] \times C^\infty[0, 1]$ and U an open subset of $L = \mathcal{L}S_{\kappa_1}^{\kappa_2}(u, v)$. Then it is a trivial consequence of (7.9) that $D \cap U \neq \emptyset$ if $U \neq \emptyset$. \square

7.11 Corollary (smooth approximation). *Let $\mathcal{U} \subset \mathcal{L}S_{\kappa_1}^{\kappa_2}(u, v)$ be open, K be a compact simplicial complex and $f: K \rightarrow \mathcal{U}$ a continuous map. Then there exists a continuous $g: K \rightarrow \mathcal{U}$ such that:*

- (i) $f \simeq g$ within \mathcal{U} .
- (ii) $g(a)$ is a smooth curve for all $a \in K$.
- (iii) All derivatives of $g(a)$ with respect to t depend continuously on $a \in K$.

Thus, the map $j: \mathcal{C}S_{\kappa_1}^{\kappa_2}(u, v) \rightarrow \mathcal{L}S_{\kappa_1}^{\kappa_2}(u, v)$ in (16) induces surjections $\pi_k(j^{-1}(\mathcal{U})) \rightarrow \pi_k(\mathcal{U})$ for all $k \in \mathbb{N}$.

Proof. Parts (i) and (ii) are exactly what was established in the first part of the proof of (7.9), in the special case where $\mathbb{L} = L^2[0, 1] \times L^2[0, 1]$, $D = C^\infty[0, 1] \times C^\infty[0, 1]$, $L = \mathcal{L}S_{\kappa_1}^{\kappa_2}(u, v)$ and $U = \mathcal{U}$. The image of the function $g = H_2: K \rightarrow \mathcal{U}$ constructed there is contained in a finite-dimensional vector subspace of D , viz., the one generated by all \tilde{v}_{ij} , so (iii) also holds. \square

7.12 Lemma ($\mathcal{C} \approx \mathcal{L}$). *Let S be complete. Then the inclusion $i: \mathcal{C}S_{\kappa_1}^{\kappa_2}(u, v)^r \rightarrow \mathcal{L}S_{\kappa_1}^{\kappa_2}(u, v)$ is a homotopy equivalence for any $r \geq 2$. Consequently, $\mathcal{C}S_{\kappa_1}^{\kappa_2}(u, v)^r$ is homeomorphic to $\mathcal{L}S_{\kappa_1}^{\kappa_2}(u, v)$.*

Proof. Let $\mathbb{L} = L^2[0, 1] \times L^2[0, 1]$, let $F = C^{r-1}[0, 1] \times C^{r-2}[0, 1]$ (where $C^k[0, 1]$ denotes the set of all C^k functions $[0, 1] \rightarrow \mathbb{R}$, with the C^k norm) and let $i: F \rightarrow \mathbb{L}$ be set inclusion. Setting $\mathcal{M} = \mathcal{L}S_{\kappa_1}^{\kappa_2}(u, v)$, we conclude from (7.8(c)) that $i: \mathcal{N} = i^{-1}(\mathcal{M}) \hookrightarrow \mathcal{M}$ is a homotopy equivalence. We claim that \mathcal{N} is homeomorphic to $\mathcal{C}S_{\kappa_1}^{\kappa_2}(u, v)^r$, where the homeomorphism is obtained by associating a pair $(\hat{\sigma}, \hat{\kappa}) \in \mathcal{N}$ to the curve γ obtained by solving (13), with σ and κ as in (14).

Suppose first that $\gamma \in \mathcal{C}S_{\kappa_1}^{\kappa_2}(u, v)^r$. Then $|\dot{\gamma}|$ (resp. κ) is a function $[0, 1] \rightarrow \mathbb{R}$ of class C^{r-1} (resp. C^{r-2}). Hence, so are $\hat{\sigma} = h \circ |\dot{\gamma}|$ and $\hat{\kappa} = h_{\kappa_1}^{\kappa_2} \circ \kappa$, since h and $h_{\kappa_1}^{\kappa_2}$ are smooth. Moreover, if $\gamma, \eta \in \mathcal{C}S_{\kappa_1}^{\kappa_2}(u, v)^r$ are close in C^r topology, then $\hat{\kappa}_\gamma$ is C^{r-2} -close to $\hat{\kappa}_\eta$ and $\hat{\sigma}_\gamma$ is C^{r-1} -close to $\hat{\sigma}_\eta$.

Conversely, if $(\hat{\sigma}, \hat{\kappa}) \in \mathcal{N}$, then $\sigma = h^{-1} \circ \hat{\sigma}$ is of class C^{r-1} and $\kappa = (h_{\kappa_1}^{\kappa_2})^{-1} \circ \hat{\kappa}$ of class C^{r-2} . Since all functions on the right side of (13) are of class (at least) C^{r-2} , the solution $\mathbf{t} = \mathbf{t}_\gamma$ to this initial value problem is of class C^{r-1} . Moreover, $\dot{\gamma} = \sigma \mathbf{t}$, hence the velocity vector of γ is seen to be of class C^{r-1} . We conclude that γ is a curve of class C^r . Further, continuous dependence on the parameters of a differential equation shows that the correspondence $(\hat{\sigma}, \hat{\kappa}) \mapsto \mathbf{t}_\gamma$ is continuous. Since γ is obtained by integrating $\sigma \mathbf{t}_\gamma$, we deduce that the map $(\hat{\sigma}, \hat{\kappa}) \mapsto \gamma$ is likewise continuous.

The last assertion of the lemma follows from (7.8(c)). \square

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